A NOTE ON DISCONTINUOUS SOLUTIONS IN THE CALCULUS OF VARIATIONS.

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IT is the object of this note to give the corner conditions and the forms of the Carathéodory Ω-function for Bliss's form of the simplest problem of the calculus of variations, and for an analogous form of the problem in space.

For the purpose of orientation and of introducing notation, a brief résumé of a part of the theory of discontinuous solutions as treated by Bolza* is given here.

1. If at a point $P_0(t_0)$ of a curve $x = x(t)$, $y = y(t)$ that minimizes or maximizes the definite integral

$$\int_{t_1}^{t_2} F[x(t), y(t), x'(t), y'(t)]dt$$

the curve possesses a corner, the corner conditions

$$F_{x'}\Big|^{t_0-0} = F_{x'}\Big|^{t_0+0}, \quad F_{y'}\Big|^{t_0-0} = F_{y'}\Big|^{t_0+0}$$

must be satisfied.

Let $P_1P_0P_2$ be an extremal (that is, a minimizing or maximizing curve) having a corner at P_0, the corner conditions being satisfied. Suppose that the continuity and other conditions usually imposed in the calculus of variations hold for the arcs P_1P_0, P_0P_2 and for the family of extremals

$$x = \phi(t, a), \quad y = \psi(t, a),$$

which contain the extremal arc $E_0 \equiv P_1P_0$ for $a = a_0$. Designate by τ_0 and $\bar{\tau}_0$ respectively the angles that the positive tangents to the arcs P_1P_0 and P_0P_2 at the point P_0 make with the positive x-axis. Then, if it is desired to find on E_a, a neighboring extremal to E_0, a point $P(t)$ and a direction $\bar{\tau}$ such that τ and τ (τ is the positive direction of E_a at P) satisfy the corner conditions, it is necessary to solve for t and $\bar{\tau}$ the equations

$$(1) \qquad F_{x'} - \bar{F}_{x'} = 0, \quad F_{y'} - \bar{F}_{y'} = 0,$$

* Bolza, Vorlesungen über Variationsrechnung, Chapter 8.

in which the arguments of $F_{x'}$, $F_{y'}$ are $\phi(t, a)$, $\psi(t, a)$, $\cos\tau(t, a)$, $\sin\tau(t, a)$, and of $\bar{F}_{x'}$, $\bar{F}_{y'}$ are $\phi(t, a)$, $\psi(t, a)$, $\cos\bar{\tau}$, $\sin\tau$. The functional determinant of (1) with respect to t, $\bar{\tau}$ at the point P_0 has the value

$$\sqrt{x'^2 + y'^2}\bar{F}_1(t_0)\Omega_0,$$

where $\Omega = \bar{F}_x \cos\tau + \bar{F}_y \sin\tau - F_x \cos\bar{\tau} - F_y \sin\tau$, and where the $\bar{F}_1$-function* has the arguments x_0, y_0, $\cos\bar{\tau}$, $\sin\bar{\tau}$. The subscript on the Ω-function denotes that the arguments ϕ, ψ, τ, $\bar{\tau}$, wherever they occur in it, have the values x_0, y_0, τ_0, $\bar{\tau}_0$ respectively. Since a usual assumption is that $F_1 \neq 0$ along an extremal, it follows that if $\Omega_0 \neq 0$ equations (1) can be solved uniquely for t and $\bar{\tau}$ and the solution $t = t(a)$, $\bar{\tau} = \bar{\tau}(a)$ will be continuous in the vicinity of $a = a_0$ and will satisfy the initial conditions $t(a_0) = t_0$, $\bar{\tau}(a_0) = \bar{\tau}_0$.

Thus can be obtained a broken extremal $E_a + \bar{E}_a \equiv P_1PP_2$ with a corner at the point P. If a is allowed to vary, a family of such extremals is obtained, and the point P describes a so-called corner curve whose parameter is a.

2. The extension of this theory to the case in which the integrand of the definite integral is a function of three variables and their derivatives is perfectly obvious and is readily accomplished, but it may be worth while to give it here.

For this case there are four equations

$$(2)\qquad \begin{gathered} F_{x'} - \bar{F}_{x'} = 0, \quad F_{y'} - \bar{F}_{y'} = 0, \quad F_{z'} - \bar{F}_{z'} = 0, \\ \cos^2\bar{\alpha} + \cos^2\bar{\beta} + \cos^2\bar{\gamma} - 1 = 0, \end{gathered}$$

the arguments of $F_{x'}$, $F_{y'}$, $F_{z'}$ being $\phi(t, u, v)$, $\psi(t, u, v)$, $\chi(t, u, v)$, $\cos\alpha(t, u, v)$, $\cos\beta(t, u, v)$, $\cos\gamma(t, u, v)$, and those of $\bar{F}_{x'}$, $\bar{F}_{y'}$, $\bar{F}_{z'}$ being $\phi(t, u, v)$, $\psi(t, u, v)$, $\chi(t, u, v)$, $\cos\bar{\alpha}$, $\cos\bar{\beta}$, $\cos\bar{\gamma}$. At a point $P_0(t_0)$ the value of the functional determinant of the equations with respect to t, $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ is found to be

$$2\sqrt{x'^2 + y'^2 + z'^2}\,\sin\alpha_0\,\sin\beta_0\,\sin\gamma_0\,\bar{F}_1(t_0)\Omega_0,$$

where

$$\Omega = \bar{F}_x \cos\alpha + \bar{F}_y \cos\beta + \bar{F}_z \cos\gamma - F_x \cos\bar{\alpha} - F_y \cos\bar{\beta} - F_z \cos\bar{\gamma}.$$

* For definition see Bolza, loc. cit., p. 196.

The $\bar{F}_1$-function* has the arguments x_0, y_0, z_0, $\cos \bar{\alpha}_0$, $\cos \bar{\beta}_0$, $\cos \bar{\gamma}_0$. If the usual assumption that it is different from zero along an extremal is made, it follows that if $\sin \bar{\alpha}_0 \sin \bar{\beta}_0 \sin \bar{\gamma}_0 \neq 0$, that is, if the extremal arc P_0P_2 does not have an initial direction which is parallel to one of the three axes, and if $\Omega_0 \neq 0$, then the system (2) can be solved uniquely for t, $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ in terms of u, v. A family of broken extremals can be determined as before, and the point P varies on a corner surface whose parameters are u, v.

3. For Bliss's form of the problem† the integral has the form

$$\int_{t_1}^{t_2} f(x, y, \tau) \sqrt{x'^2 + y'^2}\, dt,$$

where $\tau = \arc\tan y'/x'$. By the methods ordinarily employed the corner conditions that must be satisfied at a point P are found to be

$$f \cos \tau - f_\tau \sin \tau - \bar{f} \cos \tau + \bar{f}_\tau \sin \bar{\tau} = 0,$$

$$f \sin \tau + f_\tau \cos \tau - \bar{f} \sin \bar{\tau} - \bar{f}_\tau \cos \bar{\tau} = 0.$$

In these equations the parameter t, of the arguments of f, f_τ, $\bar{f}$, $\bar{f}_\tau$, has been replaced by s, the length of arc.

Their functional determinant with respect to s, $\bar{\tau}$ is

$$\bar{f}_1 \omega,$$

where

$$f_1 = f + f_{\tau\tau}, \quad \omega = \bar{f}_x \cos \tau + \bar{f}_y \sin \tau - f_x \cos \bar{\tau} - f_y \sin \bar{\tau}.$$

Since here also $f_1 \neq 0$, the properties of ω are identical with those of the corresponding function in the Weierstrass form of the problem.

4. In a paper entitled " The space problem of the calculus of variations in terms of angle," to be published in the *American Journal of Mathematics*, I have considered an integral of the form

$$\int_{t_1}^{t_2} f(x, y, \tau, \sigma) \sqrt{x'^2 + y'^2 + z'^2}\, dt,$$

* For definition see Bliss and Mason, "The properties of curves in space which minimize a definite integral," *Transactions Amer. Math. Society*, vol. 9 (1908), p. 442.

† See Bliss, "A new form of the simplest problem of the calculus of variations," *Transactions Amer. Math. Society*, vol. 8 (1907), pp. 405–414.

τ and σ being defined by the relations

$$\frac{x'}{\sqrt{x'^2 + y'^2 + z'^2}} = \cos\sigma\cos\tau,$$

$$\frac{y'}{\sqrt{x'^2 + y'^2 + z'^2}} = \cos\sigma\sin\tau,$$

$$\frac{z'}{\sqrt{x'^2 + y'^2 + z'^2}} = \sin\sigma.$$

Geometrically σ is defined as the angle that the positive tangent to a curve makes with its projection in the xy-plane, and τ is the angle that this projection makes with the positive x-axis. The corner conditions that must be satisfied by the extremals of such an integral are

$$f\cos\sigma\cos\tau - f_\tau\frac{\sin\tau}{\cos\sigma} - f_\sigma\sin\sigma\cos\tau$$
$$- \bar{f}\cos\bar{\sigma}\cos\bar{\tau} + \bar{f}_\tau\frac{\sin\bar{\tau}}{\cos\bar{\sigma}} + \bar{f}_\sigma\sin\bar{\sigma}\cos\bar{\tau} = 0,$$

$$f\cos\sigma\sin\tau + f_\tau\frac{\cos\tau}{\cos\sigma} - f_\sigma\sin\sigma\sin\tau$$
$$- \bar{f}\cos\bar{\sigma}\sin\bar{\tau} - \bar{f}_\tau\frac{\cos\bar{\tau}}{\cos\bar{\sigma}} + \bar{f}_\sigma\sin\bar{\sigma}\sin\bar{\tau} = 0,$$

$$f\sin\sigma + f_\sigma\cos\sigma - \bar{f}\sin\bar{\sigma} - \bar{f}_\sigma\cos\bar{\sigma} = 0.$$

Here again t has been replaced by s.

A somewhat lengthy piece of reckoning gives as the functional determinant of this system, with respect to s, $\bar{\tau}$, $\bar{\sigma}$,

$$-\frac{\bar{f}_1\omega}{\cos\bar{\sigma}},$$

where

$$f_1 = (f\cos^2\sigma - f_\sigma\sin\sigma\cos\sigma + f_{\tau\tau})(f + f_{\sigma\sigma})$$
$$- (f_\tau\tan\sigma + f_{\tau\sigma})^2,$$

$$\omega = \bar{f}_x\cos\sigma\cos\tau + \bar{f}_y\cos\sigma\sin\tau + \bar{f}_z\sin\sigma$$
$$- f_x\cos\bar{\sigma}\cos\bar{\tau} - f_y\cos\bar{\sigma}\sin\bar{\tau} - f_z\sin\bar{\sigma}.$$

If we make the usual assumption $f_1 \neq 0$ (we assume also that $\cos\bar{\sigma} \neq 0$), the ω-function has the same properties as the corresponding function in three-dimensional space.

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