THE TENTH REGULAR MEETING OF THE SOUTHWESTERN SECTION.

THE tenth regular meeting of the Southwestern Section was held at the University of Kansas, at Lawrence, on Saturday, December 2, 1916. About thirty persons were in attendance, among them the following twenty-three members:

Professor C. H. Ashton, Professor A. A. Bennett, Professor Henry Blumberg, Professor W. C. Brenke, Professor P. J. Daniell, Professor E. W. Davis, Professor I. M. DeLong, Professor E. P. R. Duval, Professor W. H. Garrett, Professor W. A. Harshbarger, Professor E. R. Hedrick, Professor O. D. Kellogg, Professor S. Lefschetz, Dr. G. H. Light, Mr. W. A. Luby, Professor U. G. Mitchell, Professor Mary W. Newson, Professor S. W. Reaves, Dr. Paul R. Rider, Professor W. H. Roever, Mr. L. L. Steimley, Professor E. B. Stouffer, and Professor J. N. Van der Vries.

Attending members were very hospitably entertained at a smoker on the evening before the meeting, and at lunch between the sessions, at the University Club. It was decided that the next meeting of the Section should be held at the University of Oklahoma, at Norman, on December 1, 1917. The following programme committee was appointed: Professors S. W. Reaves (chairman), Henry Blumberg, O. D. Kellogg (secretary).

The following papers were presented:

(1) Professor A. A. BENNETT: "Newton's method in general analysis."

(2) Professor S. LEFSCHETZ: "Double integrals of the second kind belonging to an algebraic surface."

(3) Professor S. LEFSCHETZ: "On the residues of double integrals belonging to an algebraic surface."

(4) Dr. P. R. RIDER: "A note on discontinuous functions in the calculus of variations."

(5) Professor W. H. ROEVER: "Series for computing the roots of the equation $x + p \tan x = 0$."

(6) Professor E. B. STOUFFER: "On the calculation of invariants and covariants of linear homogeneous differential equations."

(7) Professor P. J. DANIELL: "The Lebesgue-Stieltjes integral."

(8) Professor HENRY BLUMBERG: "An example of a discontinuous function with a certain remarkable property."

(9) Dr. G. H. LIGHT: "Note on the relation between the focal points of an extremal when both end points are free."

(10) Professor G. C. EVANS: "An extension of Hadamard's formula for a linear functional."

(11) Professor H. C. GOSSARD: "On the relations between the faces and edges of a tetrahedron."

(12) Professor HENRY BLUMBERG: "Convex functions."

(13) Professor E. R. HEDRICK: "Functions that are nearly analytic."

Professor Blumberg's first paper and Professor Gossard's paper were read by title. Abstracts of the papers follow.

1. In this paper, Professor Bennett considers the formal character of Newton's method of approximating to a root of an equation, whether algebraic or transcendental. By generalizations of the notions sum, unknown quantity, coefficient, product, absolute value, etc., Newton's method is shown to apply also in a general field whose restrictions are considered explicitly. In particular, application is made to certain types of non-linear integral equations. The paper appeared in the *Proceedings* of the National Academy of Sciences, October, 1916.

2. The plan followed by E. Picard in his development of the theory of double integrals of the second kind is somewhat as follows: (a) reduction, largely by algebraic processes, of these integrals to a normal form; (b) development of the theory of integrals of total differentials of the third kind; (c) application of (b) to a special class of double integrals of the second kind; (d) investigation of the periods of integrals in the normal form; (e) enumeration of integrals of the second kind. In this note, Professor Lefschetz shows that this theory can be much simplified by starting with d. From this a and b then follow easily, and thus most of the algebraic work found in Picard's presentation is avoided.

3. A double integral belonging to an algebraic surface F may have cyclic or polar residues. It is shown in Professor

Lefschetz's second paper that the latter have the following two properties: (*a*) those with respect to points on the same algebraic curve have a zero sum; (*b*) those with respect to a given ordinary point of F also have a zero sum. The first follows at once from a well known theorem, while by means of quadratic transformations the second is reducible to a proposition given by Poincaré. Conversely, given a set of numbers corresponding in a suitable way to an assigned set of algebraic curves of F and satisfying the conditions a and b above, there is a double integral belonging to F of which they are the polar residues. No special conditions have to be satisfied by cyclic residues. This note will appear in the *Quarterly Journal of Pure and Applied Mathematics.*

4. In this paper, Dr. Rider gives the corner conditions and the Carathéodory Ω-function for discontinuous solutions in the calculus of variations for the form of the problem considered by Bliss in several papers, and for a new form of the space problem discussed by the author himself in a paper to be published in the *American Journal of Mathematics.*

5. For computing the roots x_n $(n = 0, 1, 2, \cdots)$ of the equation $x + p \tan x = 0$, Professor Roever gives two types of series. The first is a power series in p, the coefficients in the series for x_n being polynomials in the reciprocals of the number $t_n = (2n + 1)\pi/2$. The second type is a power series in $p + 1$, the coefficients being polynomials in the reciprocals of the numbers r_n, or the nth solution of the equation $x - \tan p = 0$. As $r_0 = 0$, a special formula is needed for the root x_0 in the second type of series.

6. Consider the system of linear homogeneous differential equations

$$(A) \qquad y_i'' + \sum_{k=1}^{n} (2p_{ik}y_k' + q_{ik}y_k) = 0 \quad (i = 1, 2, \cdots, n),$$

where p_{ik} and q_{ik} are functions of the independent variable x. In a paper presented to the Society at Chicago in April, 1915, Professor Stouffer calculated the complete system of seminvariants of (A). In the present paper, he obtains the complete system of semi-covariants by the use of certain operators. He also sets up the systems of partial differential equations which must be satisfied by any invariant or covariant of (A). These equations are expressed in terms of the seminvariants and semi-covariants. It is thus easy to test whether an expression in the seminvariants and semi-covariants which may arise in the geometry of the system is also an invariant or covariant.

7. The Lebesgue-Stieltjes integral is defined by Professor Daniell in this paper by means of sets of points in a way similar to that employed in defining the Lebesgue integral. It possesses the Lebesgue property that the limit of the integral is the integral of the limit. Like the Stieltjes integral, it is discontinuous, and the "measure" of even a single point may be finite. It is therefore suitable for use in physics, particularly electricity, where charge (or mass) may be used as a "measure" of a set of points. Most of the fundamental properties of Lebesgue integrals are shown to be satisfied by it. The paper will be offered to the *Transactions*.

8. Professor Blumberg shows with the aid of Zermelo's Wohlordnungssatz that there exist real, one-valued, discontinuous functions $f(x)$, defined in the linear continuum, such that if P is a given perfect set, then $f(x)$ is discontinuous with respect to P at every point of P.

9. In the *Mathematische Annalen*, volume 75, Rasmadse has considered the case of one fixed and one variable end point. He found that the focal point t_2'' comes before the focal point t_1'' if the integral $J = \int_{t_1}^{t_2} F(x, y, x', y')$ is a minimum. Dr. Light shows that t_2'' and t_1'' must coincide in the general case in which both end points are free.

10. In a recent paper in the BULLETIN (November, 1916) Dr. Fischer gives an interesting formula for a linear functional $T[\varphi]$ which has continuity merely of order k in its argument $\varphi(x)$, instead of order zero. Hadamard's form for a linear functional, which is itself slightly more general than the others, admits, according to Professor Evans, this extension in a very simple form. If T, depending on $\varphi(x)$ in the interval (a, b), has continuity merely of order k, it may be written: $T[\varphi(x)] = \lim_{n=\infty} \int_{a-\epsilon}^{b+\epsilon} \varphi(x)\Psi(x, n)dx,$ where $\varphi(x)$ is extended

continuously with its first k derivatives beyond the interval (a, b) to the interval $(a - \epsilon, b + \epsilon)$, ϵ being a quantity arbitrarily small, but positive. For $\Psi(u, n)$ may be taken such a function as $\sqrt{n/\pi}\,T[\{1 - (u - x)^2\}^n]$. These results will probably be published in the Cambridge Colloquium Lectures.

11. By expressing the absolute in terms of the faces of a tetrahedron and also in terms of the edges, both equations being written in line coordinates, and then equating coefficients, Professor Gossard obtains (*a*) six equations of the type $(3V/2)^2 e_{12}^2 = A_{00}A_{33} - A_{03}^2$; (*b*) twelve of the type $(3V/2)^4 e_{12}^2 e_{20}^2 = (A_{03}A_{13} - A_{01}A_{33})^2 + 4A_{33}(3V/2)^4$; (*c*) three of the type $9V^2[(e_{01}^2 + e_{23}^2) - (e_{02}^2 + e_{13}^2)] = 16(A_{01}A_{23} - A_{02}A_{13})$, where V is the volume, A_{ij} the product of the areas of the faces opposite the vertices $e_i = 0$ and $e_j = 0$ by the cosine of the angle between the two faces, and where e_{ij} is the length of the edge from the vertex $e_i = 0$ to the vertex $e_j = 0$.

12. The real, one-valued function $f(x)$ of the real variable x is (according to Jensen) said to be "convex" in the interval (a, b), where it is defined if, for every pair of values x_1, x_2 of x, the inequality

$$f\left[\frac{x_1 + x_2}{2}\right] \leqq \frac{f(x_1) + f(x_2)}{2}$$

holds. Similarly, the function $f(x, y)$, defined in the square S, is said to be convex in S, if for every pair of points (x_1, y_1), (x_2, y_2) in S, the inequality

$$f\left[\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right] \leqq \frac{f(x_1, y_1) + f(x_2, y_2)}{2}$$

holds. Professor Blumberg shows that if the convex function $f(x)$ has at least one discontinuity between a and b, it is necessarily non-measurable in the Lebesgue sense. Furthermore, if the convex function $f(x, y)$, defined in the square S, is such that the set of functional values of $f(x, y)$ on the boundary of S has a finite upper bound, then $f(x, y)$ is continuous at every interior point of S. This result remains true if for "square" we substitute "closed Jordan curve," and, in fact, more general curves may be allowed. Generalizations to n dimensions are given.

13. In this paper Professor Hedrick describes a method of characterizing functions of a complex variable that are not analytic, but that differ only slightly from analytic functions. For example, the function $\psi(z) = f(z) + \epsilon\varphi(z)$, where $f(z)$ is analytic, and ϵ is a real number, will differ but little from $f(z)$, if ϵ is chosen small, at all points where $|\varphi(z)|$ is finite. The Riemann surface for such functions may be studied by means of its relation to that for $f(z)$. A series of examples of this type and of more general types is given.

O. D. KELLOGG,
Secretary of the Section.

THE RELATIONS OF MATHEMATICS TO THE NATURAL SCIENCES.

PRESIDENTIAL ADDRESS DELIVERED BEFORE THE AMERICAN MATHEMATICAL SOCIETY, DECEMBER 28, 1916.

BY PRESIDENT E. W. BROWN.

THE duty enjoined on the President of the American Mathematical Society of delivering an address at the close of his term of office is at once an opportunity and a danger. It is one of the rare occasions when he is able to discuss matters which are unsuited for a memoir and when it is proper to try to take a somewhat broader view of his subject than is suggested in investigations designed to elucidate some special part of it. In so doing, he necessarily must look into the future and attempt to foresee it through such indications of the present as may seem significant; and the danger of becoming a false prophet or of raising an unnecessary alarm is unattractive to anyone, least of all to those who have the lifelong habit of feeling their way into the unknown by roads slowly constructed and securely laid. I am willing to run this risk because I believe that there are certain matters connected with the future of mathematical science which need fuller consideration than they have received of late years. In discussing them, one must necessarily tread on debatable ground. It is, however, a happy custom to regard the matter contained in a presidential address not as an official presen-