BULLETIN OF THE

AMERICAN MATHEMATICAL SOCIETY.

THE TWENTY-SECOND SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE twenty-second summer meeting of the Society was held at the University of California on Tuesday, and at Stanford University on Wednesday, August 3–4, 1915, in connection with the Panama-Pacific International Exposition. The following thirty-four members of the Society were in attendance upon the two sessions:

Professor R. E. Allardice, Dr. Nathan Altshiller, Dr. Charlotte C. Barnum, Dr. A. A. Bennett, Dr. B. A. Bernstein, Professor H. F. Blichfeldt, Professor C. E. Brooks, Dr. Thomas Buck, Professor E. F. A. Carey, Professor E. E. De Cou, Dr. H. B. Curtis, Professor G. C. Edwards, Dr. Elizabeth B. Grennan, Professor F. L. Griffin, Professor F. W. Hanawalt, Professor Charles Haseman, Professor M. W. Haskell, Professor E. R. Hedrick, Professor L. M. Hoskins, Dr. Frank Irwin, Professor C. J. Keyser, Professor D. N. Lehmer, Professor J. H. McDonald, Professor W. A. Manning, Professor H. C. Moreno, Professor R. E. Moritz, Professor L. I. Neikirk, Professor C. A. Noble, Professor E. W. Ponzer, Mr. F. D. Posey, Professor T. M. Putnam, Professor H. W. Stager, Professor H. W. Tyler, Professor S. E. Urner.

Professor M. W. Haskell, chairman of the San Francisco Section, presided at the session on Tuesday afternoon, and Professor R. E. Allardice at that on Wednesday afternoon.

Tuesday morning was devoted to a joint session with the American Astronomical Society and Section A of the American Association for the Advancement of Science. Addresses were delivered by Professors C. J. Keyser on "The human significance of mathematics," and G. E. Hale on "The work of a modern observatory." The astronomers, mathematicians, and physicists lunched at the Faculty Club as guests of Professors Leuschner, Haskell, and E. P. Lewis.

The social programme included a dinner with the American Astronomical Society at the Hotel Oakland on Wednesday evening, an excursion to the Lick Observatory on Friday, and a luncheon given by Mrs. Phoebe Hearst at the Hacienda del Pozo de Verona on Saturday.

The following papers were presented at this meeting:

(1) Professor L. E. DICKSON: "Invariantive classification of pairs of conics modulo 2."

(2) Professor C. J. DE LA VALLÉE POUSSIN: "Sur l'intégrale de Lebesgue."

(3) Mr. A. R. SCHWEITZER: "On the solution of a class of functional equations."

(4) Dr. NATHAN ALTSHILLER: "On the circles of Apollonius."

(5) Dr. DUNHAM JACKSON: "Proof of a theorem of Haskins."

(6) Dr. W. W. KÜSTERMANN: "Fourier constants of functions of two variables."

(7) Dr. B. A. BERNSTEIN: "A set of four independent postulates for Boolean algebras."

(8) Professor G. A. MILLER: "Limits of the degree of transitivity of substitution groups."

(9) Professor R. D. CARMICHAEL: "On the representation of numbers in the form $x^3 + y^3 + z^3 - 3xyz$."

(10) Professor H. S. WHITE: "Seven points on a gauche cubic curve."

(11) Professor M. W. HASKELL: "The del Pezzo quintic curve."

(12) Professor L. J. RICHARDSON: "A phase of Roman mathematics."

(13) Dr. C. A. FISCHER: "Functions of surfaces with exceptional points or curves."

(14) Mr. A. R. WILLIAMS: "On a birational transformation connected with a pencil of cubics."

(15) Professor F. N. COLE: "Note on the triad systems in 15 letters."

(16) Professor A. B. COBLE: "The determination of the lines on a cubic surface."

(17) Mr. H. S. VANDIVER: "An aspect of the linear congruence, with applications to the theory of Fermat's quotient."

(18) Dr. C. H. FORSYTH: "An interpolation formula based upon central and multiple differences."

(19) Dr. G. M. Green: "On isothermally conjugate nets of space curves."

(20) Professor L. P. Eisenhart: "Surfaces of rolling and transformations of Ribaucour."

(21) Mr. A. R. Schweitzer: "Generalized quasi-transitive functional relations."

(22) Professor L. M. Hoskins: "'Quantity of matter' in dynamics."

(23) Dr. A. A. Bennett: "The iteration of functions of one variable."

Professor Richardson was introduced by Professor Haskell and Mr. Williams by Professor Lehmer. Professor de la Vallée Poussin's paper was communicated to the Society through Professor Birkhoff. Professor White's paper was read by Professor Haskell. The papers by Professor Dickson, Professor de la Vallée Poussin, Mr. Schweitzer, Dr. Jackson, Dr. Küstermann, Professor Miller, Professor Carmichael, Dr. Fischer, Professor Cole, Professor Coble, Mr. Vandiver, Dr. Forsyth, Dr. Green, and Professor Eisenhart were read by title.

Abstracts of the papers, including Professor Keyser's address, follow below. The abstracts are numbered to correspond to the papers in the list above.

In Professor Keyser's address the treatment, conducted in the spirit appropriate to an international exposition, aims at being interesting and intelligible to the general educated public. A sketch of the manner in which a historian of mathematics might hope to vindicate the science is followed by a more elaborate exposition of the task of attaining the same end through an account of the present state of the science. The claims of mathematics to human regard as based on its applications in a wide range of other sciences and arts are next dealt with. This matter is followed by a characterization of the modern critical movement of the science and of the resulting conception of the distinctive character of mathematics, especially in its relation to modern developments in logic. The chief emphasis of the address falls on the bearings of the science as distinguished from its applications—upon the significance of mathematics for man regarded as finding his supreme interest in seeking permanent values in an inpermanent world.

1. With a conic F modulo 2 is associated covariantively a point A, called its apex, and a unique line L, and conversely A and L uniquely determine F (Madison Colloquium Lectures, 1914, page 69). Hence the projective classification of pairs of conics F and F' is equivalent to that of the systems A, L, A', L' of two points and two lines and the degenerate systems in which one or more of the four elements are absent. A simple geometrical discussion of such systems leads Professor Dickson to the theorem: Two pairs of conics modulo 2 are projectively equivalent if and only if they have the same properties as regards existence of apices and covariant lines, distinctness of apices and lines, and incidence of apices and lines. These properties are expressed analytically by very simple modular invariants, which therefore form a fundamental system of modular invariants of two conics.

2. Professor de la Vallée Poussin's paper, which reproduces a part of his recent course of lectures at Harvard University, will appear in the *Transactions*.

3. The object of Mr. Schweitzer's paper is to discuss the solution of the equations

$$f\{\lambda_1 f(t_1, t_2, \cdots, t_n, x_1), \lambda_2 f(t_1, t_2, \cdots, t_n, x_2), \cdots,$$
$$\lambda_{n+1} f(t_1, t_2, \cdots, t_n, x_{n+1})\} = \mu f\{x_{i_1}, x_{i_2}, \cdots, x_{i_{n+1}}\},$$

where $n = 1, 2, 3$, etc.; the subscripts $i_1, i_2, \cdots, i_{n+1}$ are distinct and range respectively over the values $1, 2, \cdots, n+1$; λ_i and μ denote functions of a single variable. The $(n+1)!$ functional equations thus defined may be put into $(1, 1)$ relation with the substitutions on $n+1$ symbols and therefore conveniently denoted by the notation

$$E\left\{\begin{pmatrix} 1 & 2 & 3 & \cdots & (n+1) \\ i_1 & i_2 & i_3 & \cdots & i_{n+1} \end{pmatrix}; \lambda_1(x), \lambda_2(x), \cdots, \lambda_{n+1}(x), \mu(x)\right\}.$$

For $n = 1$ the equations have been completely discussed by the author in previous articles. For $n > 1$ the special instance of identity of some or all of the functions $\lambda_i(x)$, $\mu(x)$ is considered. In the simplest case, namely, when $n = 2$ and the functions $\lambda_i(x)$, $\mu(x)$ are all identical and equal to x, the following theorems on the equations

$$E\left\{\begin{pmatrix} 1 & 2 & 3 & \cdots & (n+1) \\ i_1 & i_2 & i_3 & \cdots & i_{n+1} \end{pmatrix}\right\}$$

are valid:

1. $E\{(1)\ (2)\ (3)\}$ implies the existence of $\psi(x)$ such that $f(x_1, x_2, x_3) = \psi^{-1}\{[\psi(x_3) - \psi(x_2)] + \theta[\psi(x_2) - \psi(x_1)]\}$, where $\theta(x)$ is arbitrary.
2. $E\{(1)\ (23)\}$ implies $f(x_1, x_2, x_3) = \psi^{-1}\{-2\psi(x_1) + \psi(x_2) + \psi(x_3)\}$.
3. $E\{(2)\ (13)\}$ implies $f(x_1, x_2, x_3) = \psi^{-1}\{\psi(x_1) - 2\psi(x_2) + \psi(x_3)\}$.
4. $E\{(3)\ (12)\}$ implies $f(x_1, x_2, x_3) = \psi^{-1}\{[\psi(x_3) - \psi(x_2)] + \theta[\psi(x_2) - \psi(x_1)]\}$, $\theta(x) = x + \theta(-x)$.
5. $E\{(123)\}$ implies $f(x_1, x_2, x_3) = \psi^{-1}\{\psi(x_1) + c\psi(x_2) + c^2\psi(x_3)\}$, where $1 + c + c^2 = 0$.
6. $E\{(132)\}$ implies $f(x_1, x_2, x_3) = \psi^{-1}\{c\psi(x_1) + \psi(x_2) + c^2\psi(x_3)\}$, where $1 + c + c^2 = 0$.

Solutions 2–4 are special cases of the Solution 1.

4. In his contribution to the modern geometry of the triangle Dr. Altshiller, starting with the usual definition of the circles of Apollonius connected with a triangle, derives, in a purely synthetic way, a series of properties, partly known, involving the Lemoine point, the symmedian lines, the Brocard diameter, etc., and leads up to the theorem: The center of any one Apollonian circle is a center of similitude of the two other circles of Apollonius, the second center of similitude being the pole of the Brocard diameter with respect to the first circle. This paper will appear in the *American Mathematical Monthly.*

5. It is a theorem due to Haskins that if f and φ are two bounded functions such that the definite integral of f^n over an interval is equal to that of φ^n for all positive integral values of n, the set of points where $\alpha \leqq f \leqq \beta$ and the set where $\alpha \leqq \varphi \leqq \beta$ have the same measure, for any pair of numbers α, β. Dr. Jackson gives a short proof of this theorem, based on a polynomial approximation and Lebesgue's theorem on the integration of a uniformly bounded sequence. The result is of particular interest in the case where the functions are monotone.

6. In volume 57 of the *Mathematische Annalen* A. Hurwitz has shown how to express the product of two ordinary Fourier

series in form of another Fourier series, or, stated differently, how to compute the Fourier constants of the product of two functions from those of the functions themselves. Dr. Küstermann solves the analogous problem for double Fourier series. The solution depends vitally upon the proof of the relation

$$\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}[f(x, y)]^2 dx dy = \sum_{\mu,\,\nu=0}^{\infty}\frac{(a^2+b^2+c^2+d^2)_{\mu\nu}}{2^{E[1/(\mu+1)]+E[1/(\nu+1)]}},$$

where $(a, b, c, d)_{\mu\nu}$ are the Fourier constants of $f(x, y)$ presumed to be bounded and integrable in the Riemann sense. The analogue of this relation for functions of a single variable is due to Parseval and was proved by him under certain restrictions on the nature of convergence of the Fourier series involved. In 1893 de la Vallée Poussin gave a proof requiring merely that the function and its square be integrable. Hurwitz, in 1903, again called attention to the wide scope of the theorem, gave a new proof, and applied it to the problem mentioned. More recently the formula has gained in general interest through the researches of Riesz and Fischer (Riesz-Fischer theorem). In the present paper the proof is established by means of Poisson's double integral.

7. The most economical set of postulates for the logic of classes is that presented by H. M. Sheffer in the *Transactions* of October, 1913. Using as basis the operation of "rejection," which Dr. Sheffer used, Dr. Bernstein obtains a set of four postulates from which the former's set of five can be deduced. The paper will appear in the *Transactions*.

8. The main theorem established by Professor Miller in the present paper is as follows: A substitution group of degree $n = kp + r$, where p is a prime number and r and k are positive integers such that $p > k$ and $r > k$, cannot be more than r times transitive unless either this group includes the alternating group of degree n or $k = 1$ and $r = 2$. By means of this theorem it can easily be proved that a group which does not include the alternating group of its degree cannot be as much as 15-fold transitive unless its degree exceeds 1,000. It is also proved that whenever $n > 12$ the theorem will always give a smaller upper limit for the degree of transitivity of a substitution group which does not include the

alternating group of its degree than the limit obtained by using the well-known formula $\frac{1}{3}n + 1$. The present paper is an extension of results obtained by the same writer in an article published in the BULLETIN, volume 4 (1898), page 140, and it aims to replace the formula $\frac{1}{3}n + 1$ for the upper limit of the degree of transitivity of the groups in question by a theorem which gives at least as low limits when $n > 7$, and gives always a lower limit when $n > 12$.

9. Among the theorems presented by Professor Carmichael in this note are the following: Every prime number other than 3 can be represented in one way and in only one way in the form $x^3 + y^3 + z^3 - 3xyz$, where x, y, z are non-negative integers; primes of the form $6n + 1$ have one additional representation in which at least one of the integers x, y, z is negative.

10. In previous theories seven points on a twisted cubic curve have been studied only in combinations that were symmetric; and the only theorem known, beyond what may be considered as definition of the curve, states that the 35 points where the seven osculating planes meet must lie on a surface of order five. Professor White examines sets of seven derived points corresponding to a triad system on seven elements, and finds them to lie upon a second cubic curve, one of thirty determined by the first seven points. Analytic proofs are given.

11. Professor Haskell's paper contains the following items: Reduction of the Hessian of the plane quintic with five cusps, showing that the five cusps and five inflexions lie on a cubic, and that the inflexions are rational in terms of the cusps. Determination of a conic with regard to which the quintic is self-dual, of a Cremona transformation of the quintic into itself, and of sets of conics and cubics connected with the cusps and inflexions.

12. Professor Richardson's paper gives a practical illustration of manual multiplication as practised by the Romans, together with a brief historical survey of this system of reckoning.

13. In a paper published in the July, 1914, number of the *American Journal of Mathematics*, Dr. Fischer has given a

definition of the derivative of a function of a surface analogous to Volterra's definition of the derivative of a function of a line, and has proved that if the derivative is continuous and approached uniformly, the first variation of the function is equal to the double integral of the derivative multiplied by the first variation of the dependent variable. In the present paper this theory is extended to the case where there are points or curves where the derivative does not exist. The theory of exceptional points is used in finding the second variation of a function of a double integral, and the theory of exceptional curves is applied to the variable boundary problem in the calculus of variations.

14. The theorem that all the cubics through eight points pass through a ninth, suggests the following transformation which is evidently birational: Fix seven of the eight points, and make correspond to a variable eighth the ninth point through which all the cubics pass.

Dr. Hart (see *Cambridge and Dublin Mathematical Journal* volume 6, page 181) has given a synthetic construction for the ninth point determined by eight given points. By an analytic treatment of this construction Mr. Williams has found the following relation between the coordinates, x', y', z', of 9 and the coordinates of 8: $x' = C_1C_2K_1$; $y' = C_2C_3K_2$; $z' = C_3C_1K_3$, where the C's are cubic functions and the K's are quadratic. These equations define a Cremona transformation.

In the above transformation there are infinitely many invariant points, all of which lie on a sextic curve of deficiency three, having double points at each of the seven fixed points. At each of these double points the tangents to the sextic coincide with the tangents to the nodal cubic having the same double point and passing through the other six fixed points. The sextic also meets the line determined by any two of these seven points in the points where this line meets the conic determined by the other five. From this follows the remarkable theorem concerning seven arbitrary points in the plane: if the line joining any two of them be cut by the conic determined by the other five, the forty-two points thus obtained lie on a sextic. This sextic satisfies seventy-seven conditions, almost three times the number necessary to determine it.

By introducing relations among the seven given points there result other types of Cremona transformations with their corresponding invariant curves. Thus if three of the seven points are on a line, we have a transformation of the seventh degree with an invariant quintic curve. And this quintic satisfies fifty-three conditions. Similarly, if six of the seven points lie on a conic, there results a quartic transformation with an invariant quartic curve fulfilling forty-seven conditions. If six points are on a conic and at the same time three are on a line, a cubic transformation results with an invariant cubic curve fulfilling twenty-nine conditions. If six points are on a conic and if at the same time three are on one straight line and three on another, the transformation is quadratic, and the invariant curve is a conic which fulfills sixteen conditions. This quadratic transformation is of a type of which inversion with respect to the unit circle is a special case.

15. Professor Cole has found, by detailed examination of the various cases, that every triad system in 15 letters, with a single exception, presents an interlacing, $(xab)(xcd)$, $(yac)(ybd)$, the exceptional case being one of Heffter's systems. It is also found that every triad system in 15 letters presents either a hexad, $(xab)(xcd)(xef)$, $(yac)(yde)(ybf)$, or triple tetrads, $(xab)(xcd)(xef)(xgh)(xij)(xkl)$, $(yac)(ybd)(yeg)(yfh)$ $(yik)(yjl)$.

16. In earlier reports Professor Coble has discussed the Cremona group $G_{6,2}$ in Σ_4 of order 51840 determined by a P_6^2 (six points in a plane). It is the purpose of this paper to show that $G_{6,2}$ furnishes the direct algebraic connection between the cubic surface C^3, mapped on the plane by cubic curves on P_6^2, and the collineation groups which arise in the trisection of the theta functions of genus 2. This connection is made by the use of irrational invariants of C^3 which are associated with the separation of the lines of the surface. The discriminant of C^3 consists of 36 irrational factors corresponding to the 36 double sixes of C^3. By forming products of 9 such factors properly selected, a set of 40 irrational "complex invariants" is obtained. These determine in Σ_4 a set of 40 quintic spreads conjugate under $G_{6,2}$. The spreads however lie in a linear system of dimension 9 and one can

choose a linearly independent set, $X_0, \cdots, X_4$; $U_0, \cdots, U_4$, with the striking property developed below.

On the other hand Klein had discovered and Burkhardt had developed a collineation group in S_4 with variables $Y_0, \cdots, Y_4$ of order 25920. If to this group there be added the correlations which transform it into itself there is obtained a mixed group g_{51840} in the contragredient variables $Y_0, \cdots, Y_4$; $V_0, \cdots, V_4$, which is isomorphic with $G_{6,2}$ and which has the following property: The variables Y, V are transformed under g_{51840} precisely as the irrational invariants X, U above are transformed under $G_{6,2}$.

This property leads to the rational determination of the lines of a cubic surface (after the adjunction of the square root of its discriminant) in terms of the solution of the form problem of Burkhardt's group without passing beyond the domain of the irrational invariants of the given surface. The methods hitherto suggested for this purpose required the introduction of an equation of degree 27 (or a resolvent of some other degree), i. e., the introduction of a binary field necessarily extraneous to the surface itself.

17. In 1903, Professor G. D. Birkhoff communicated to Mr. Vandiver the following theorem:

If p is a prime integer and a is a positive integer prime to p, then there is at least one and not more than two sets (x, y) such that

$$a \equiv \pm x/y \pmod{p},$$

where x and y are integers prime to each other and $0 < x < \sqrt{p}$, $0 < y < \sqrt{p}$.

In the present paper a proof of this theorem is given, involving a continued fraction algorithm for the direct determination of each set. It is also shown that the fact that there exists at least one set (x, y) follows from a theorem due to Minkowski. The least positive residues of the integers

$$1^{p-1}, \quad 2^{p-1}, \cdots, \quad (p-1)^{p-1}$$

modulo p^2 are termed *proper residues* modulo p^2. By an extension of the result mentioned above the following theorem is derived:

There are not more than

$$p - \tfrac{1}{2}(1 + \sqrt{2p - 5})$$

and not less than $[\sqrt{p}]$ incongruent proper residues modulo p^2, if p is a prime > 2.

Applications to problems in connection with Fermat's last theorem are then discussed.

18. The accuracy of an interpolation based upon finite differences is increased as the values of the independent variable occurring in the differences approach the value of the independent variable of the ordinate which is interpolated. Dr. Forsyth has derived an interpolation formula wherein these values are identical.

Furthermore, instead of the usual averaging of differences (i. e., use of first differences) to supply differences which are missing (as always happens in central differences), differences are used whose orders are in keeping with the order of the interpolation formula itself.

19. In Dr. Green's note is given a new geometric interpretation of Bianchi's condition for isothermally conjugate nets of curves on a surface. It will appear in the *Proceedings of the National Academy of Sciences.*

20. In a note published in volume 23 of the *Rendiconti dei Lincei* Bianchi has defined as a surface of rolling the surface described by a point invariably fixed with respect to a surface S_0 as the latter rolls over an applicable surface S, which Bianchi calls the surface of support. He shows that, given any surface Σ, the problem of finding pairs of applicable surfaces S_0 and S such that Σ is a surface of rolling as S_0 rolls over S reduces to the integration of a partial differential equation of the second order and of a Riccati equation. Two surfaces are said to be in the relation of a transformation of Ribaucour when they constitute the envelope of a two-parameter family of spheres such that the lines of curvature correspond on the two surfaces, corresponding points being on the same sphere. Professor Eisenhart has shown that the necessary and sufficient condition that either surface of two so related be a surface of rolling with the locus of the centers of the spheres for surface of support is that the two surfaces be isothermic and in the relation of a so-called transformation D_m, discovered by Darboux.

21. As a quasi-transitive functional relation in the most general sense, Mr. Schweitzer defines

$$\phi\{f_1(t_1, t_2, \cdots, t_n, x_1), f_2(t_1, t_2, \cdots, t_n, x_2), \cdots, \\ f_{n+1}(t_1, t_2, \cdots, t_n, x_{n+1})\} = \psi(x_1, x_2, \cdots, x_{n+1}),$$

where $n = 1, 2, 3$, etc. From this class of equations other classes are deduced, first, by the homologous transposition of the x's in the left hand member and, second, by substituting x's for some or all of the t's in the set of equations thus obtained (including the original class). For example, one obtains by homologous transposition of the x's when $n = 2$

$$\phi\{f_1(x_1, t_1, t_2), f_2(x_2, t_1, t_2), f_3(x_3, t_1, t_2)\} = \psi(x_1, x_2, x_3)$$
$$\phi\{f_1(t_1, x_1, t_2), f_2(t_1, x_2, t_2), f_3(t_1, x_3, t_2)\} = \psi(x_1, x_2, x_3).$$

Interesting problems are presented by systems of equations belonging to the same or different classes.

22. The proposition that the accelerations of different bodies under the action of equal forces are inversely proportional to their masses is often asserted to be merely a definition of mass. The object of the paper by Professor Hoskins is to show that, in applying this proposition, our interpretation of it involves the notion of mass as a quantitative measure of the matter of which the bodies are composed.

23. Dr. Bennett considers several topics connected with the iteration of functions of one variable. Matrices with an infinite number of elements are used to obtain a classification of the types of series which are to be distinguished with respect to the nature of their iteration. The different types are considered, with particular reference to questions of convergence. The iteration of functions of a real variable is also considered. The paper will appear in the *Annals of Mathematics.*

THOMAS BUCK,
Acting Secretary.

GROUPLESS TRIAD SYSTEMS ON FIFTEEN ELEMENTS.

BY DR. LOUISE D. CUMMINGS AND PROFESSOR H. S. WHITE.

(Read before the American Mathematical Society April 24, 1915.)

FROM previous publications and a paper presented to this Society in October, 1914, 44 different triad systems on 15 elements (Δ_{15}) are known. These 44 systems separate into two types 23: of the systems each contain one or more systems