AN ELEMENTARY DOUBLE INEQUALITY FOR THE ROOTS OF AN ALGEBRAIC EQUATION HAVING GREATEST ABSOLUTE VALUE.

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LET there be given an algebraic equation of the nth degree written

$$x^n + C_{n,1}a_1x^{n-1} + C_{n,2}a_2x^{n-2} + \cdots + a_n = 0,$$

where $C_{n,1}$, $C_{n,2}$, $\cdots$ denote binomial coefficients. Let x_1, x_2, $\cdots$, x_n denote its roots, and X the greatest absolute value of any of these roots.

From the equations which express the coefficients in terms of the roots

$$C_{n,k}a_k = (-1)^k\Sigma x_1x_2\cdots x_k \quad (k = 1, 2, \cdots n)$$

we infer at once that

$$| a_k | \leqq X^k.$$

In fact there are precisely $C_{n,k}$ terms on the right-hand side of the kth equation, each of which is not greater than X^k.

Hence, if α stands for the greatest of the quantities

$$| a_1 |, \ | a_2 |^{1/2}, \ \cdots, \ | a_n |^{1/n},$$

it is clear that we have $X \geqq \alpha$.

Also from the given algebraic equation we obtain directly

$$| x |^n \leqq C_{n,1} | a_1 | \, | x |^{n-1} + \cdots + | a_n |,$$

where x represents any of the quantities x_1, x_2, $\cdots$, x_n. Replacing $| x |$ by one of its possible values X, and each quantity $| a_k |$ by the quantity α^k, at least as great, we obtain

$$| X |^n \leqq C_{n,1}\alpha \, | X |^{n-1} + \cdots + \alpha^n,$$

which may be written

$$X^n \leqq (X + \alpha)^n - X^n.$$

Transposing the term X^n to the left-hand side and extracting the nth root of the two members of the resulting inequality, we find

$$X \leqq \frac{\alpha}{2^{1/n} - 1}.$$

The roots of largest absolute value X are restricted by the double inequality

$$\alpha \leqq X \leqq \frac{\alpha}{2^{1/n} - 1},$$

where α denotes the largest of the quantities

$$| a_1 |, | a_2 |^{1/2}, \cdots, | a_n |^{1/n}.$$

The inequality $X \geqq \alpha$ was noted by R. D. Carmichael and T. E. Mason,* who observed also that the lower limit is reached if the equation is

$$(x + \alpha)^n = 0.$$

It is also evident that the upper limit found above is reached if the equation is

$$2x^n - (x + \alpha)^n = 0.$$

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CERTAIN NON-ENUMERABLE SETS OF INFINITE PERMUTATIONS.

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1. THE simplest element of a permutation is the pairing of one of the objects permuted with a number indicating its place in the permutation. Such a pairing may be called a *primitive element* and denoted by (i, n), where n is the object and i the number of the place assigned to it. In this paper the objects will all be numbers, finite or transfinite.

2. Permutations of finite sets are simply collocations of primitive elements. They are conveniently denoted by ex-

* BULLETIN, vol. 21 (1914), pp. 14–22; in particular p. 20.