THE TWENTY-FIRST ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE twenty-first annual meeting of the Society was held in New York City on Friday and Saturday, January 1–2, 1915. The attendance at the four sessions included the following ninety-five members:

Mr. J. W. Alexander, II, Dr. E. S. Allen, Professor G. N. Armstrong, Professor C. S. Atchison, Professor Clara L. Bacon, Mr. A. A. Bennett, Professor G. D. Birkhoff, Professor Joseph Bowden, Professor J. W. Bradshaw, Professor E. W. Brown, Dr. T. H. Brown, Dr. R. W. Burgess, Professor B. H. Camp, Professor C. W. Cobb, Professor A. B. Coble, Dr. Emily Coddington, Professor A. Cohen, Professor F. N. Cole, Dr. G. M. Conwell, Professor J. L. Coolidge, Professor Elizabeth B. Cowley, Professor G. H. Cresse, Professor L. P. Eisenhart, Professor G. C. Evans, Mr. G. W. Evans, Professor F. C. Ferry, Professor H. B. Fine, Dr. C. A. Fischer, Professor W. B. Fite, Professor T. M. Focke, Professor A. B. Frizell, Professor O. E. Glenn, Professor J. W. Glover, Dr. G. M. Green, Professor T. H. Gronwall, Professor C. C. Grove, Professor J. G. Hardy, Professor C. N. Haskins, Professor H. E. Hawkes, Professor E. R. Hedrick, Dr. A. A. Himowich, Professor T. F. Holgate, Professor L. A. Howland, Professor E. V. Huntington, Dr. Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Dr. J. K. Lamond, Professor O. C. Lester, Professor Florence P. Lewis, Mr. P. H. Linehan, Professor W. R. Longley, Professor A. C. Lunn, Professor James Maclay, Professor Mansfield Merriman, Dr. E. J. Miles, Professor H. H. Mitchell, Dr. R. L. Moore, Dr. F. M. Morgan, Professor Frank Morley, Professor F. R. Moulton, Mr. G. W. Mullins, Professor G. D. Olds, Professor W. F. Osgood, Dr. Alexander Pell, Professor Anna J. Pell, Dr. G. A. Pfeiffer, Professor A. D. Pitcher, Professor Arthur Ranum, Dr. H. W. Reddick, Professor R. G. D. Richardson, Professor H. L. Rietz, Dr. R. B. Robbins, Professor E. D. Roe, Jr., Dr. Joseph Rosenbaum, Dr. Caroline E. Seely, Professor L. P. Siceloff, Dr. L. L. Silverman, Professor H. E. Slaught, Professor P. F. Smith, Professor Virgil Snyder, Professor K. D. Swartzel, Professor Elijah Swift, Professor Evan Thomas, Professor H. W. Tyler, Mr. H. S. Vandiver, Professor E. B. Van Vleck, Professor Oswald Veblen, Mr. J. N. Vedder, Mr. H. E. Webb, Professor C. J. West, Professor H. S. White, Professor E. E. Whitford, Professor J. W. Young.

The President of the Society, Professor E. B. Van Vleck, occupied the chair at the opening session, being relieved by Vice-President L. P. Eisenhart and after the annual election by the President-elect, Professor E. W. Brown, and Vice-President Oswald Veblen. The council announced the election of the following persons to membership in the Society: Dr. Florence E. Allen, University of Wisconsin; Dr. Nathan Altshiller, University of Washington; Dr. D. F. Barrow, University of Texas; Dr. R. B. Robbins, Sheffield Scientific School; Mr. C. H. Yeaton, University of Chicago. Fifteen applications for membership in the Society were received.

The total membership of the Society is now 709, including 69 life members. The total attendance of members at all meetings of the past year was 428; the number of papers read was 192. The number of members attending at least one meeting during the year was 255. At the annual election 210 votes were cast. The Treasurer's report shows a balance of \$9,461.75, including the life membership fund of \$5,198.85. Sales of the Society's publications during the year amounted to \$1,843.67. The Library now contains about 5,100 bound volumes.

The annual meeting this year was especially marked as the occasion of the delivery of President Van Vleck's retiring address, the subject of which was "The rôle of the point set theory in geometry and dynamics." The address was given on Friday afternoon before an audience of nearly one hundred, a considerable number having come on from the West and from the American Association meeting at Philadelphia.

Another special feature of the meeting, on the social side, was the annual dinner and smoker held at the Yale Club on Friday evening with an attendance of seventy members.

At the annual election, which closed on Saturday morning, the following officers and other members of the Council were chosen:

President,	Professor E. W. Brown.
Vice-Presidents,	Professor F. R. Moulton,
	Professor Oswald Veblen.

Secretary, Professor F. N. COLE.
Treasurer, Professor J. H. TANNER.
Librarian, Professor D. E. SMITH.

Committee of Publication,
Professor F. N. COLE,
Professor VIRGIL SNYDER,
Professor J. W. YOUNG.

Members of the Council to Serve until December, 1917,
Professor G. D. BIRKHOFF, Professor R. G. D. RICHARDSON,
Professor O. E. GLENN, Professor W. H. ROEVER.

Committees were appointed to consider the question of holding a colloquium in connection with the summer meeting at San Francisco, and the possible relations of the Society to the field now covered by the *American Mathematical Monthly.* A committee is also considering matters relating to the teaching of mathematics in the schools.

The following papers were read at this meeting:

(1) Professor L. P. EISENHART: "Transformations of surfaces Ω."

(2) Dr. L. L. SILVERMAN: "On the notion of summability for the limit of a function of a continuous variable."

(3) Professor A. B. COBLE: "A configuration in finite geometry."

(4) Professor A. B. COBLE: "The elliptic norm curve in S_4."

(5) Professor J. E. ROWE: "The symbolic and actual form of certain combinants of two binary n-ics."

(6) Professor ARTHUR RANUM: "On the differential geometry of the cyclic (circled) surfaces."

(7) Professor A. B. FRIZELL: "An enumeration of integral algebraic polynomials."

(8) Dr. DUNHAM JACKSON: "Expansion problems with irregular boundary conditions."

(9) Dr. G. M. GREEN: "Hypersurfaces and families of curves defined by solutions of a partial differential equation of the second order."

(10) Dr. CAROLINE E. SEELY: "Certain non-linear integral equations."

(11) President E. B. VAN VLECK, Presidential address: "The rôle of the point set theory in geometry and dynamics."

(12) Professor W. F. OSGOOD: "On the division of space of n dimensions by a simple closed manifold."

(13) Professor G. D. BIRKHOFF: "The functions of several variables defined by linear difference and differential equations."

(14) Professor G. D. BIRKHOFF: "Note on the reducibility of maps."

(15) Professors VIRGIL SNYDER and F. R. SHARPE: "Certain quartic surfaces belonging to infinite discontinuous cremonian groups."

(16) Mr. H. S. VANDIVER: "A property of cyclotomic integers and its relation to Fermat's last theorem."

(17) Professor J. L. COOLIDGE: "Circular transformations and complex space."

(18) Professor G. A. MILLER: "Note on several theorems due to A. Capelli."

(19) Professor EDWARD KASNER: "The generalized concept of differential element."

(20) Professor F. N. COLE: "Note on solvable quintics."

(21) Professor G. C. EVANS: "Note on the variation of a function depending on all the values of another function."

(22) Professor E. V. HUNTINGTON: "A set of postulates for elementary dynamics" (preliminary communication).

(23) Professor F. R. MOULTON: "The solution of an infinite system of implicit functions, with an application to Hill's lunar theory."

(24) Professor C. N. HASKINS: "On the roots of the incomplete gamma function."

(25) Dr. W. C. GRAUSTEIN: "On the geodesics and geodesic circles on a developable surface."

(26) Dr. D. F. BARROW: "Oriented circles in space."

(27) Professor JAMES MACLAY: "A transformation of polynomials relative to the exponents."

(28) Mr. J. W. ALEXANDER, II: "A method for resolving the singularities of algebraic manifolds."

(29) Professor T. H. GRONWALL: "On the summation method of de la Vallée-Poussin."

(30) Professor T. H. GRONWALL: "An integral equation of the Volterra type."

(31) Professor T. H. GRONWALL: "On the distortion in conformal representation."

The papers of Professor Rowe, Dr. Seely, Mr. Vandiver, Professor Miller, Professor Cole, Dr. Graustein, Dr. Barrow, and the last two papers of Professor Gronwall were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In the October number of the *Transactions* Professor Eisenhart showed that by an adaptation of the theorem of Moutard for differential equations of the Laplace type with equal invariants it is possible to transform a surface S, referred to a conjugate system with equal point invariants, into a surface S_1 upon which the parametric curves are of the same sort. This transformation, which he called a transformation K, is such that for the congruence of lines joining corresponding points on S and S_1 the developables cut these surfaces in the parametric system and the focal points on any line are harmonic with respect to the points of S and S_1 on the line. In the present memoir he considers the particular case where this congruence is normal. Surfaces orthogonal to such a congruence have been considered by Demoulin, who called them surfaces Ω. A surface S which admits a transform S_1 such that the congruence is normal is characterized by the property that the conjugate system with equal invariants is 2, 0, to use the notation of Guichard. When such a surface is given, the determination of the normal congruence is a question of quadratures. With the aid of the theorem of permutability of transformations K applied to these surfaces S, it is possible to establish transformations A of surfaces Ω into surfaces Ω such that a surface and a transform are the envelope of a two-parameter family of spheres with the lines of curvature corresponding on the two surfaces. These transformations also admit a theorem of permutability.

Isothermic surfaces are surfaces Ω, and the transformations A which give rise to new isothermic surfaces are the transformations D_m, discovered by Darboux and studied at length by Bianchi. Surfaces with isothermal spherical representation of their lines of curvature are surfaces Ω with one of the surfaces S at infinity, and consequently require special study. When the surfaces resulting from such a surface by transformations A are surfaces of the same sort, the transformations are the same as those established by the author some years ago from a different point of view.

2. A sequence of numbers may be regarded as a function $u(n)$ of a discrete variable n. Toeplitz has defined a related sequence

$$v(n) = \sum_{i=1}^{i=n} k(n, i)u(i),$$

where $k(n, i)$ is a function of two discrete variables, and has obtained conditions on $k(n, i)$ which are necessary and sufficient that the existence of the limit of $u(n)$ imply the existence of the limit of $v(n)$ and the equality of the two limits. Replacing the discrete variables by continuous variables and the sign of summation by an integral sign, Dr. Silverman considers the related functions $u(x)$ and

$$v(x) = \int_a^x k(x, y)u(y)dy$$

and obtains conditions on $k(x, y)$ which are necessary and sufficient that the existence of the limit of $u(x)$ imply the existence of the limit of $v(x)$ and the equality of the two limits. The theorem obtained applies in particular to improper integrals.

3. For $p = 1, 2, 3$ there are sets of even theta functions in p variables which are related to a set of $2p + 2$ points in a projective space S_p. The purpose of Professor Coble's first paper is to show that, in the finite geometry associated with the functions, there exist for any value of p configurations which are in tactical correspondence with such a set of points.

4. Among all the elliptic norm curves, that of order five in S_4 is peculiar in that the quadrics which contain the curve serve to define projectively a Cremona transformation. This transformation is studied in Professor Coble's second paper as an aid to the location of the point sets mentioned in the first paper.

5. The important rôle played by the combinants of two binary forms in the theory of rational plane curves is dependent upon the fact that if they are expressed in *combinant form* (i. e., in terms of the two-rowed determinants of the matrix of coefficients of the two binary forms) an easy translation scheme makes it possible to transform these combinants into the equations of covariant curves of rational plane curves. The purpose of Professor Rowe's paper is to derive a series of combinants of two binary n-ics which are not only

in combinant form, but which may be expressed as determinants of comparatively low order whose constituents are functions of the two-rowed determinants of the matrix of coefficients of the two binary forms; also, it is shown how these may be represented symbolically in such a way that the actual expression for a combinant of any two particular n-ics may be written out with facility. Possibly the simplest combinant of this type is the one whose vanishing is the necessary and sufficient condition that two binary n-ics have polar cubics with two common factors.

6. In the study of cyclic surfaces much attention has been given in the past to certain very special kinds, for instance to anallagmatic surfaces, including cyclides. In the theory of the general cyclic surface, very little progress has been made, and the few results obtained have been confined almost exclusively to metrical and conformal properties. The more fundamental descriptive properties, which are invariant under the 24-parameter group of sphere transformations, have hardly been touched. They form the subject of the present paper by Professor Ranum, which is based on two earlier papers on the projective differential geometry of n-space, one published in the *Annali* in 1912 and the other soon to appear in the *American Journal.* The results are obtained by employing the well-known correspondence between the projective line geometry of 4-space and the descriptive circle geometry of 3-space. It is shown that from this viewpoint cyclic surfaces naturally fall into eleven classes, of which four are annular and seven non-annular. Among the striking results is the fact that there exists on every non-annular surface (excepting only those which are anallagmatic) a certain fundamental pair of curves, whose importance for the surface is somewhat similar to that of the edge of regression for a developable surface.

7. By arranging the natural numbers as products of powers of primes, Professor Frizell makes an actual enumeration of the algebraic equations in which all coefficients are positive integers. Similarly the equations in which negative integers occur as coefficients are put into one-to-one correspondence with the set of positive fractions in their lowest terms. Then the two sets of equations may be assigned to the even and odd numbers respectively.

8. In most of the expansion problems which have been studied in detail, in connection with ordinary linear differential equations, the boundary conditions are of the sort which Birkhoff calls *regular*. A notable exception occurs in the case of certain series obtained by Liouville (*Journal de Mathématiques*, volume 3) from differential equations of order higher than the second, with boundary conditions of a particular type. It is readily seen that these conditions are irregular, and consequently the question of the convergence of the series, which was left open by Liouville, also falls outside the scope of Birkhoff's general treatment. Dr. Jackson shows that the difference between this case and those usually studied is an essential one; expansions which would be uniformly convergent, if the analogy of the ordinary theory were sustained, are found actually to diverge with great rapidity. The difference is explained by the fact that the characteristic functions, instead of being essentially of the nature of trigonometric functions, as in the familiar cases, involve real exponential factors which fundamentally affect their properties. The analysis is applicable not only to Liouville's boundary conditions, but also to a considerably more general class of conditions in which these are included.

9. Segre has pointed out that on a surface (two-spread) defined by solutions of a partial differential equation of the second order in any space of higher dimensionality than three, there exists one and only one conjugate net of curves, in the sense that the tangents to curves of one family of the net, at the points where these curves meet a fixed curve of the other family, form a developable surface. It is well known also that with every one-parameter family of curves on a surface in three-space may be associated a unique conjugate family. Dr. Green's paper has two purposes in view: first, to study a surface in a space of n dimensions referred to a conjugate net of parameter curves; and second, to provide a substitute for conjugacy on a surface when $n > 3$, in order to be able to associate with a one-parameter family of curves on such a surface a second family uniquely determined by the first. The quasi-conjugacy defined in the paper is a reciprocal relation, and Segre's unique conjugate net plays the same rôle as does the asymptotic net on a surface in ordinary space, in the sense that each family of the conjugate net is quasi-

conjugate to itself. The analysis of the paper consists in a study not of a single partial differential equation of the form

$$Ay_{uu} + By_{uv} + Cy_{vv} + Dy_u + Ey_v + Fy = 0,$$

but of a completely integrable system of two equations of which the above is one. Wilczynski's methods are used in obtaining the invariants and covariants of the configurations considered.

10. The non-linear integral equations discussed in Miss Seely's paper are of the type susceptible of treatment by Volterra's algebra of permutable and non-permutable functions. Volterra has proved that the binomial quadratic

$$\int_a^b \xi(s, \tau)\xi(\tau, t)d\tau + 2\xi(s, t) + \lambda k(s, t) = 0$$

is satisfied by a certain power series in λ within its circle of convergence, the coefficients being symbolic powers of the known kernel $k(st)$ except for constant factors. Miss Seely proves that the equation is also satisfied by the analytic prolongation of this series at every non-singular point in the λ plane, and shows that the singular points are identical with the characteristic constants of the kernel $k(st)$ as given by the Fredholm theory. If λ_h is a pole of order $m > 1$ of the Fredholm resolvent kernel $k(st, \lambda)$, it is an infinity of order $(2m - 3)/2$ of the solution of the binomial quadratic; if $m = 1$, it is a branch point of first order. An expression is given for the solution valid at every point in its star. Extension is made to binomial integral equations of nth degree, and to the general quadratic and cubic, and some discussion is given of the question of the existence of solutions at the singular points.

11. President Van Vleck's retiring address will appear in full in the BULLETIN for April.

12. Let a simple closed surface in space of three dimensions, which is made up of a finite number of pieces of analytic surfaces meeting one another along a finite number of analytic curves, be denoted as an analytic polyhedron. The corresponding $(n - 1)$-dimensional manifold in space of n dimensions shall be called an analytic hyperpolyhedron. Professor

Osgood proves the theorem that such a manifold divides the space in which it lies into an interior and an exterior region. The method consists in slicing by a family of parallel planes or hyperplanes and considering the regions into which one of them is divided.

13. Professor Birkhoff's first paper contains an extension of the methods of an earlier paper* to the treatment of certain classes of functions of several complex variables, and will appear in the *Acta Mathematica*.

14. In this paper Professor Birkhoff carries further the principle of "reducibility of maps" employed earlier by him,† and thus limits still further the form of "completely reduced" maps. The importance of the notion of reducibility is that it appears probably to afford an adequate although extremely complex method for the solution of the four-color problem. This paper will be offered to the *American Journal of Mathematics*.

15. The paper of Professors Snyder and Sharpe has for its purpose the establishment of two theorems:

Theorem I. The quartic subjected to the single condition of passing through a non-hyperelliptic sextic curve of genus three is invariant under an infinite discontinuous group of birational transformations.

Theorem II. The transformations of the infinite discontinuous group under which the most general quartic surface passing through a sextic curve of genus two remains invariant can be expressed in terms of cremonian transformations.

In connection with the first theorem the equation of the surface is derived and the equations of the transformations are determined; it is shown that the transformations are cremonian and non-involutorial and that no transformations exist other than those obtained. It is believed that this surface is the first illustration of one which possesses an infinite discontinuous group, but contains neither a pencil of elliptic curves nor a net of hyperelliptic curves of genus two. The

* "The generalized Riemann problem for linear differential equations and the allied problems for linear difference and q-difference equations," *Proceedings of the American Academy of Arts and Sciences*, vol. 49 (1913), pp. 521–568.

† *American Journal of Mathematics*, vol. 35 (1913), pp. 115–128.

equation of the surface (Fano surface) mentioned in the second theorem is found, and also the equations of two involutorial space transformations, which generate the infinite group.

16. By extending some methods due to Kummer (Berlin *Sitzungsberichte,* 1857) Mr. Vandiver is enabled to set up a relation from which the criteria of Kummer (Mirimanoff, *Crelle,* volume 128), Legendre, and Furtwängler (*Wiener Sitzungsberichte,* 1912, page 589) in reference to Fermat's last theorem may be immediately deduced.

17. In the *Mathematische Annalen,* volume 22 (1883), Stéphanos establishes a one-to-one correspondence between the projectivities of the binary domain and the points of a projective S_3. In Professor Coolidge's paper these formulas are adapted to establishing a one-to-one correspondence between the points of a complex projective S_3 and the real circular transformations of the Gauss plane. The correspondence is used on the one hand to give real representation of various loci in complex space, and on the other to discover new theorems concerning circular transformations.

18. The useful theorem that every subgroup of a group of order p^m, p being any prime number, is invariant under operators of the group which are not contained in this subgroup is commonly supposed to be due to G. Frobenius and to W. Burnside. Cf. "Encyclopédie des Sciences mathématiques" tome I, volume 1, page 608; W. Burnside, "Theory of Groups of Finite Order," 1911, page 122. As a matter of fact, this theorem is evidently a special case of a theorem which was proved eleven years earlier by A. Capelli, and which may be stated as follows: A necessary and sufficient condition that every subgroup of a group G is transformed into itself by at least one operator which is not contained in this subgroup is that G contains no more than one Sylow subgroup of any given order.*

This theorem is clearly equivalent to the theorem that a necessary and sufficient condition that every subgroup of a group G appears in at least one of its possible series of com-

* A. Capelli, *Atti R. Accademia dei Lincei, Memorie*, (3), vol. 19 (1884), p. 272.

position is that G is the direct product of its Sylow subgroups; which the second edition of Pascal's Repertorium der höheren Mathematik, volume 1, 1910, page 199, accredits to W. Burnside. The object of Professor Miller's paper is to direct attention to this and to other theorems proved by A. Capelli, and to extend one of these theorems. In particular, the operators of any composite group G are arranged in the usual rectangular form with respect to an invariant subgroup H as the first row, and the operators of the first column are so selected as to generate the smallest possible group. The cross-cut of this group and H cannot contain more than one operator, or more than one subgroup, in any complete set of conjugates when this is also a complete set of conjugates under G. In particular, this cross-cut is the direct product of its Sylow subgroups. The smallest group above mentioned must be G itself whenever G is cyclic and H does not include any Sylow subgroup of G.

19. An ordinary differential element in the plane may be represented by a point and a parabolic curve of the nth order passing through it. An infinite power series in x, convergent or divergent, Professor Kasner regards as defining a differential element of infinite order. This may be pictured most concretely, in general, by the broken line running from the given point to the successive centers of curvature. If further we allow series with fractional exponents, we obtain, in addition to regular elements, new types of irregular elements. According to the nature of the coefficients in the series, the element may be real or imaginary. In earlier papers the author has classified elements with respect to the conformal group. The present paper gives the classification of all elements (real or imaginary, regular or irregular, convergent or divergent) in metric geometry.

20. Professor Cole obtains a set of very simple resolvent equations for the algebraic solution of quintics of the form $x^5 + \alpha x + \beta = 0$ when such solution is possible. This paper will appear in a later number of the BULLETIN.

21. In order to express the variation of a function depending continuously on all the values of another function between a and b, in terms of the integral of the derivative of the de-

pendent function with respect to the functional argument multiplied by the variation of that argument, Volterra finds it necessary to make use of four postulates, two of which are statements of uniform conditions. In the present paper Professor Evans seeks to reduce the number of these assumptions. He proves that if the functional derivative exists for every point of a continuous curve $\varphi(x)$ between $x = \xi_1$ and $x = \xi_2$, and takes the values α at ξ_1 and β at ξ_2, then between ξ_1 and ξ_2 it takes all the values between α and β. If the functional derivative of the function F exists for all values of ξ between $x = a$ and $x = b$, and for all functions φ in a certain region, being continuous with respect to φ in that region, and if F vanishes for φ_1 and φ_2, $\varphi_2 - \varphi_1$ being everywhere a positive quantity, then the derivative of F vanishes at some point of one of the curves of the pencil $(\varphi_1\varphi_2)$. If the derivative is continuous in regard to its functional argument in the neighborhood of $\varphi(x)$, and finite, and integrable along $\varphi(x)$ in regard to its other argument ξ from a to b, then the first variation may be written in the desired form. Moreover if in addition, the derivative is continuous in regard to ξ, so that two of Volterra's postulates are established, the other two follow.

22. Professor Huntington's paper contains a tentative set of postulates for the composition and resolution of forces, expressed in terms of the following fundamental concepts: (1) a class of elements $a, b, c, \cdots$, to be called forces, or vectors; (2) a relation $a \uparrow b$, to be called like-directedness; a relation $a \equiv b$, to be called congruence; and a rule of combination $a \oplus b = c$, to be called vector addition.

23. The infinite system of equations considered by Professor Moulton may be written in the form

$$\begin{aligned} x_1 &= f_1(x_1, x_2, \cdots; m) = f_1^{(0)} + f_1^{(2)} + \cdots, \\ x_2 &= f_2(x_1, x_2, \cdots; m) = f_2^{(0)} + f_2^{(2)} + \cdots, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

where m is an arbitrary parameter, $x_1, x_2, \cdots$ an infinite system of dependent variables, $f_i^{(j)}$ the totality of terms of f_i which are homogeneous of degree j in $x_1, x_2, \cdots$. One set of hypotheses under which the conclusion follows is:

(*a*) The $f_i^{(0)}$ vanish for $m = 0$.
(*b*) The $f_i^{(0)}$ and the coefficients of the $f_i^{(j)}$ are regular analytic functions of m for $|m| \leqq \rho$.
(*c*) There exists a finite constant M_i and a series $c_1x_1+c_2x_2 + \cdots = s$, where the c_i are regular functions of m for $|m| \leqq \rho$, such that s converges if $|m| < \rho$, $|x_i| < r_i$ and M_is^j dominates $f_i^{(j)}$.
(*d*) $M_i \leqq r_i$ if $|m| \leqq \rho' \leqq \rho$, $|x_i| \leqq r_r' \leqq r_i$.
Under these conditions there exist series of the form

$$x_i = a_i^{(1)}m + a_i^{(2)}m^2 + \cdots$$

which satisfy the original equations identically in m and converge if $|m| < \sigma > 0$.

By an application of these results, suitably modified, the expansions as power series in m of the Fourier coefficients of the expressions for the coordinates in Hill's variational orbit can be shown to converge for $|m|$ sufficiently small. This ocmpletes the proof of the convergence, in sufficiently restricted domains, of all the series Hill employed in his lunar theory.

24. The location of the real roots of the function $P(x)$ defined by

$$P(x) \equiv \frac{\Sigma(-1)^s}{s!}\frac{1}{x+s}$$

has been studied by Bouquet who found that in each of the intervals $-2n < x < -2n + \frac{1}{2}$, $-2n + \frac{1}{2} < x < -2n + 1$ there lies *at least* one root, while outside these intervals there are no real roots. Professor Haskins shows by an application of the theorem of Budan-Fourier that in each of these intervals there is *exactly* one root.

25. If $x_i = x_i(s)$, $(i = 1, 2, 3)$, is the parametric representation (in terms of the arc) of an arbitrary but fixed geodesic of a developable surface y, then, as is well known,

$$y_i = x_i - a\left(\frac{R}{T}\alpha_i - \gamma_i\right) \qquad (i = 1, 2, 3),$$

where α and γ are the directions of the tangent and binormal at the general point of the curve x, R and T are the radii of curvature and torsion at this point, and a is a parameter. Dr.

Graustein finds as the finite equation of the geodesics on y

$$\left(c_2 + \frac{R}{T}c_3\right)a - (c_1 + c_3 s) = 0,$$

where c_1, c_2, c_3 are arbitrary constants; and as the finite equation of the geodesic circles on y

$$\left(\frac{a}{r} - k_1\right)^2 + \left(\frac{1}{r}\left(s - a\frac{R}{T}\right) - k_2\right)^2 = 1,$$

where r is the radius of geodesic curvature of the general circle and k_1, k_2 are arbitrary constants. These equations afford simple deductions of facts already known and lead to a number of results which seem to be new.

26. Dr. Barrow considers the subject of oriented circles in three dimensions. He makes use of the Plücker coordinates of a circle which are formed from the pentaspherical coordinates of any two spheres intersecting in the circle; and accomplishes the orientation by means of a redundant coordinate defined by a quadratic relation as a two-valued function of the Plücker coordinates. A geometric interpretation for a certain invariant function of the coordinates of two oriented circles is given; and a few theorems are proved about systems of oriented circles whose coordinates satisfy one or more linear relations.

27. The transformation considered by Professor Maclay consists in replacing the exponents of a polynomial by their positive residues, in particular their least positive residues, relative to an arbitrary modulus, and forming the difference of the two polynomials. The effect of successive applications of the transformation and the relations of the resultant polynomials are studied.

28. Mr. Alexander's paper contains a simple method for transforming any algebraic surface birationally into a surface without singularities lying in a space of sufficiently many dimensions. By a direct generalization, the device may be used to resolve the singularities of any n dimensional manifold.

Let $f(x_0, x_1, x_2, x_3) = 0$ be an algebraic surface. The transformation

$$x_i = \frac{\partial f}{\partial x_i} \qquad (i = 0, 1, 2, 3) \tag{1}$$

transforms f into its dual $F(x_0, x_1, x_2, x_3) = 0$. The transformation

$$\xi_{ij} = x_i x_j, \tag{2}$$

together with (1), transforms f into a surface ϕ in 15-dimensional space. It will be found that every singular point p of f which is not a uniplanar point is transformed into a curve l of ϕ and that the multiplicity of each point of l is lower than that of p. A uniplanar point may be simplified by a transformation (2′) analogous to (2). By means of a finite number of transformations (2) or (2′) followed in each case by a projection of the surface back into 3-space, all the singularities may be resolved.

29. Letting $u_0 + u_1 + \cdots + u_n + \cdots$ be any series, de la Vallée-Poussin (*Bulletin de l'Académie Belgique,* 1908) introduced the expressions

$$V_n = \sum_{v=0}^{n} \frac{n!n!}{(n-v)!(n+v)!} u_v,$$

and the original series is summable when $\lim_{n=\infty} V_n$ exists. In the present paper, Professor Gronwall forms the generating identity of these sums, viz., writing $\xi = 4x/(1+x)^2$,

$$\sum_{n=0}^{\infty} a_n x^n = \frac{1-x}{1+x} \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} n! n!} V_n \xi^n.$$

From this identity, a generalization of Abel's theorem for power series is obtained; it is shown that any series summable by Cesàro's method is also summable by the present method with the same sum; and various other properties of the sums V_n are proved. The theory may be generalized to the sums obtained from the fundamental identity by making

$$\xi = 1 - \left(\frac{1-x}{1+x}\right)^{2m}.$$

30. Professor Gronwall gives a direct proof of the existence and uniqueness of a bounded solution of the integral equation

$$\varphi(x, y) = f(x, y) + \int_0^x A(x, y, s)\varphi(s, y)ds$$
$$+ \int_0^y B(x, y, t)\varphi(x, t)dt + \int_0^x ds \int_0^y C(x, y, s, t)\varphi(s, t)dt,$$

when f, A, B and C are bounded and have their discontinuities regularly distributed in the rectangle $0 \leqq x \leqq a$, $0 \leqq y \leqq b$. This proof avoids the usual decomposition of the rectangle (a, b) into smaller rectangles. An application to partial differential equations of the hyperbolic type is given.

31. If the power series $f(x) = x + a_2x^2 + \cdots + a_nx^n + \cdots$ is such that $z = f(x)$ defines the conformal representation of the unit circle in the x-plane on a simple region in the z-plane, then, as Koebe has shown, there exist upper and lower boundaries for $|f(x)|$ and $|f'(x)|$ when $|x| = r < 1$, these boundaries being independent of the coefficients of $f(x)$.

Professor Gronwall obtains the following approximations to these boundaries:

$$\frac{2}{9}r < |f(x)| < \frac{3r}{(1-r)^2}, \quad \frac{1}{5}\cdot\frac{1-r}{(1+r)^3} < |f'(x)| < 5\cdot\frac{1+r}{(1-r)^3}.$$

These approximations are of the correct order of magnitude, as is shown by the example $f(x) = x/(1-x)^2$.

F. N. Cole,
Secretary.

THE WINTER MEETING OF THE SOCIETY AT CHICAGO.

The thirty-fourth regular meeting of the Chicago Section of the American Mathematical Society was held at the University of Chicago on Monday and Tuesday, December 28–29, 1914, it being the third regular meeting of the Society at Chicago. Eighty persons were in attendance upon the sessions, including the following sixty-four members of the Society:

Professor R. P. Baker, Professor W. H. Bates, Mr. William Betz, Professor G. A. Bliss, Professor Henry Blumberg, Professor Daniel Buchanan, Professor H. E. Buchanan, Dr. Josephine Burns, Professor W. H. Bussey, Professor W. D. Cairns, Professor R. D. Carmichael, Professor A. F. Carpenter, Dr. E. H. Clarke, Professor H. E. Cobb, Professor D. R. Curtiss, Dr. W. W. Denton, Professor L. E. Dickson, Mr. C. R. Dines, Professor L. W. Dowling, Professor Arnold Dresden, Professor Arnold Emch, Professor A. B. Frizell, Dr. M. G. Gaba,