NOTE ON REMOVABLE SINGULARITIES.

BY MR. W. E. MILNE.

IN his dissertation* Kistler proves the following theorem: "*Let $f(z_1, \cdots, z_n)$ be analytic throughout the neighborhood of a point $(a_1, \cdots, a_n)$ with the exception at most of the points of a finite number of analytic manifolds, each of which is at most $(2n-4)$-dimensional. Then it is analytic in the excepted points also, if properly defined there.*"

A similar theorem is given below. The two theorems differ in two respects. (1) Kistler's theorem requires the excepted points to lie on analytic manifolds, while the present one does not; (2) the present theorem requires that for *every*† pair of variables the excepted locus reduce to isolated points when the remaining $n-2$ variables are fixed, while Kistler's theorem requires this to hold simply for *one* pair. But it is to be noted that the hypotheses of the present theorem will in general‡ be fulfilled if the singular manifolds are analytic.

The theorem is as follows:

THEOREM. *Let the function $\varphi(z_1, \cdots, z_n)$ of n complex variables, $n > 2$, be analytic in the region $(S_1, \cdots, S_n)$ except for the points of a $(2n-4)$-dimensional locus of such character that when any $n-2$ of the variables are given any fixed values in their respective regions, and z_i and z_j alone vary, then the singularities occur only at isolated points (a_i, a_j) in (S_i, S_j). Under these conditions φ has a limit in the points of the singular locus, and if defined as equal to its limit, will be analytic without exception in $(S_1, \cdots, S_n)$.*

Proof: Let $z_3, \cdots, z_n$ be held fast at $(a_3, \cdots, a_n)$ any point of $(S_3, \cdots, S_n)$, and consider φ as a function of z_1 and z_2. From the hypothesis we see that singularities can occur only at isolated points of (S_1, S_2). But isolated singularities for a function of two complex variables are removable; so φ, if properly defined, will be analytic in z_1 and z_2 throughout (S_1, S_2).

* "Ueber Funktionen von mehreren komplexen Veränderlichen," Basel, 1905. § 7.

† Evidently all that is really necessary is that with each variable z_i it be possible to associate another z_j such that if the remaining $n-2$ are fixed, the singular points in z_i, z_j are isolated.

‡ I hope shortly to be able to show that the excepted cases can always be avoided by a suitable linear transformation of the independent variables, and hence that the words "in general" can be replaced by "always."

Give to z_2 any value a_2 in S_2, and φ will be analytic in z_1 alone. This holds for every choice of the fixed values assigned to $z_2, \cdots, z_n$. In a similar manner we find φ analytic in each remaining variable alone.

Now apply the theorem of Hartogs* which states that if a function of n complex variables is analytic in each one separately, it is analytic in all n variables taken together. Hence φ is analytic throughout $(S_1, \cdots, S_n)$.

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CONCERNING A CERTAIN TOTALLY DISCONTINUOUS FUNCTION.

BY PROFESSOR K. P. WILLIAMS.

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ONE of the most important properties of a continuous function is that it actually assumes every value between any two of its values. It is well known that a function can, however, possess this property without being continuous. An actual example to illustrate this seems to have been first given by Darboux in 1875. A function that is sometimes cited in this connection is due to Mansion.† The function that the latter gives actually takes all values between any two, but is discontinuous at the single point $x = 0$. Functions of this sort can be easily constructed by arbitrarily assigning the values at certain points, according to the function concept of Dirichlet. More interest would therefore attach to such a function if it is given by one and the same expression throughout its region of definition. The function given by Mansion does not, however, possess this property; for it contains the function $E(x)$, defined, as in number theory, as the integer equal to, or next smaller than x.

The purpose of this note is to give a function that takes every value between 0 and 1 inclusive, when x varies over the closed interval (0, 1), but which is discontinuous at every point. This function will, furthermore, be represented by one and the same analytical expression throughout its whole region of definition.

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* *Math. Ann.*, vol. 62 (1905), p. 1.

† "Continuité au sens analytique et continuité au sens vulgaire," in *Mathesis*, 1899.