OSGOOD'S THEORY OF FUNCTIONS.

Lehrbuch der Funktionentheorie. Von DR. W. F. OSGOOD. Erster Band mit 158 Figuren. Zweite Auflage. (Bd. XX: 1, B. G. Teubner's Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften.) B. G. Teubner, 1912. 8vo. xii + 766 pp.

OSGOOD'S Theory of Functions is a monument of American scholarship. Comparatively few mathematical works of broad, *comprehensive* character have as yet been written by Americans. While there is a superabundance of good or indifferent texts in English on elementary topics such as analytic geometry and elementary calculus, we lack a sufficiency on more advanced subjects. The appearance of a work such as this, realizing the highest standards for breadth and accuracy of scholarship, is an event. It is therefore to be soberly regretted that it has not been published in the English tongue. Written by a man of English name and of true American stock, it appears in the language of that country which needs it least. Yet while I voice a widely expressed regret that Osgood's treatise is not written in English, it must, on the other hand, be recognized that mathematics is international; further, that the inspiration and spirit of the work is in good part German.

Inasmuch as no previous review has appeared in the BULLETIN,* the task of the reviewer should be a double one, not merely to consider the changes occurring in the new edition but also to survey the work as a whole, even though the characteristics and essential features are by this time well known to many readers. The attempt to do this, however, brings despair to the reviewer because of the many-sidedness of the work and its exceeding richness of content. It is amazing what a wealth of ideas and detail is packed into a single volume. The judiciousness of selection and thorough assimilation of material combine to make the work par excellence the most authoritative treatise upon the theory of analytic functions.

I.

The first edition appeared in 1907 and was divided into three Abschnitte. The first of these collects the "Theorems and

* See, however, a translation of the preface by H. S. White, vol. 13, p. 398.

methods of the function theory of real variables" and the concepts and results of the Mengenlehre which are indispensable for a thorough and modern treatment of analytic functions. In so doing the author implicitly recognizes—what has in truth become increasingly apparent with mathematical progress—that a course in real analysis should logically precede a study of the function theory of the complex variable, unless the two theories are interwoven as in French treatises on analysis. This is nearly, if not quite, imperative when the Riemann-Cauchy standpoint is adopted. Though Osgood departs outwardly from the French plan in organizing and separating the function theories of the real and of the complex variable, the dependence of the latter upon the former is made so intimate as to produce essentially the same effect as does their interweaving. After the way has been cleared in Abschnitt I, the elements and principal working theorems of the theory of analytic functions are developed in Abschnitt II with great rapidity. Abschnitt III treats of the applications, which are given in successive chapters on elliptic functions, Reihen- und Produktenentwickelungen, the elementary functions, and logarithmic potential.

In the new edition there is an expansion from the previous 642 pages to 766 pages. The last chapter of Abschnitt III has now been split off and amplified into a fourth Abschnitt containing two chapters, which are devoted to logarithmic potential and to conformal representation and the uniformization of analytic functions respectively. A complete treatment of the last subject constitutes the main addition to the first edition and accounts approximately for a third of the increase in the size of the volume. Of this addition I shall speak further on.

A second change of much importance is the insertion of numerous footnotes concerning the authorship of the theorems and results, their origin, place of publication, etc. The compilation of these footnotes must have been laborious, but they are invaluable and greatly enhance the usefulness of the work, especially for the teacher and investigator.

Beside these major changes numberless minor changes are made from page to page in the new edition, in consequence of which the revision may be called an extensive one, though with practically no change of plan. Not infrequently extensions of theorems are added which result from placing less restriction upon the hypotheses. Some of the more striking changes in their order of occurrence are as follows.

At the close of chapter 1 the familiar Heine-Borel theorem is appended under the heading: Ein allgemeines Theorem. The author's conception of it is interesting. To quote his words, when in the vicinity of every point of a closed interval a certain property has been established and the continuation of the proof consists in the application of the method of "Einschachtelung der Intervalle," the last part of the proof can be formulated in the aforesaid theorem. The appropriateness of the usual designation of the theorem is not discussed. Also, so far as I have observed, no use is anywhere made of the theorem. This may be regretted, since only through examples will the student gain an idea of its mode of application and of its frequent great convenience.

A noteworthy modification occurs in the treatment of Goursat's theorem that when $f(z)$ has a derivative at every point of a region, this derivative must be continuous. In the first edition Goursat's original proof by the integral method was given, and this was followed by Moore's proof by the differential method. These are now replaced by a beautiful demonstration of the author's own construction (page 349), based on an extension of the theorem of Morera that a function $f(z)$ which is continuous within a given region will be analytic if its integral taken around an arbitrary closed curve within the region is always zero. This extension, which is found only in the new edition, was suggested to Osgood by a similar idea of Bôcher in the treatment of logarithmic potential. It replaces the requirement that $\int f(z)dz$ shall vanish for a contour of arbitrary shape by the less restrictive requirement that it shall vanish when taken around rectangles in the region whose sides are parallel to the real and imaginary axes. This change is an important one for simplicity of application. The theorem occupies a rather prominent place in Osgood's development of the function theory. By its aid Weierstrass' theorem concerning the analytic character of the sum of a uniformly convergent series of analytic functions is established, and likewise the unlimited differentiability of the series term by term.

As specimen minor additions in the new edition I may cite a new and desirable section (II 8, § 7) relating to the six anharmonic ratios and the corresponding division of the complex plane; a brief treatment (II 12, § 8) of the functional equation

$$f(x+y) = \frac{f(x) + f(y)}{1 - f(x)f(y)}$$

for tan x; Painlevé's theorem (page 53) on the continuity of the boundary values of a function continuous within a region S; and Osgood's illuminating example of non-uniform convergence (pages 87–89). The section on the convergence of infinite products of functions (II 11, § 7) has been much modified, Bôcher's definition of uniform convergence being now replaced by the more usual Weierstrassian definition.

I turn now to the consideration of the new paragraphs on the uniformization of analytic functions. This subject has naturally a special attraction for the author, not only on account of its intrinsic importance but also because it is at bottom a question of conformal representation, to whose solution he had already made a most vital contribution. The first edition closed with a brief paragraph upon the uniformization of analytic functions. Nearly simultaneously with its appearance the almost complete uniformization was effected by Poincaré and Koebe independently. The inclusion of this was the one thing absolutely necessary to bring the last chapter up to date. This has been effected by adding the new matter at the end without material modification of the plan. In consequence, it seems to me rather difficult for the reader to catch the uniting thread of thought which runs through the chapter. His attention is shifted rather rapidly from one thing to another, though at the close the connection becomes clear. Notwithstanding, the chapter is one of the most fascinating in the entire volume. For this reason, as well as to aid anyone who may wish to skim the chapter superficially, I shall analyze its contents further.

To a considerable degree the arrangement is parallel with historical development. The author starts the chapter by presenting Riemann's problem of building a simply connected piece of a plane, with or without continuous and one-to-one correspondence of boundary, conformally upon a closed or open circle respectively. It is announced that first of all the problem is to be considered for a *closed* region with a regular boundary; in other words, for a "region S." But the very first theorem states that by means of a Green's function the interior T of *any* simply connected piece of a plane whose boundary consists of more than one point can be built conformally upon the interior of a circle. In fact, it is the problem of the conformal representation of *interiors* which is the key note of the new chapter. Next, the construction of a Green's function for a region S is made to depend upon the solution of the boundary value problem, and this is attained by the Murphy-Neumann-Schwarz alternating process. Returning then to the consideration of open regions, we have next Osgood's own great achievement, the proof of the existence of a Green's function for any simply connected piece T of the character specified above. The proof is applicable, no matter how evil the character of the boundary and irrespective of its possession of inaccessible points. A few words here concerning possible complications of the boundary and the extraordinary things which may happen when account is taken of what becomes of the boundary points in the representation, would not only have been illuminating but would have made visible the difficulty of the theorem.

Next follow paragraphs upon the circular triangle with zero-angles (where we consider again the imaging of a closed region) and upon Picard's theorem. Then the consideration of the uniformization of analytic functions is begun with a novel construction of a simply-connected Riemann surface for any algebraic function. This surface has, in general, an infinite number of leaves, though there is a coordinate case of closure ($p = 0$) which is put to one side by the author for subsequent treatment in a "Nachtrag." It is this surface which, as it presently appears, solves the uniformization problem by its conformal* representation upon the whole or a part of the plane. The surface is introduced to the reader in a paragraph headed: "Der algebraische Fall, Uniformisierung vermöge automorpher Funktionen mit Grenzkreis." But automorphic functions with a limit-circle are not introduced till the close of the next paragraph and are used in only one of the three cases possible for the uniformization. The following sections contain several references (pages 721, 723, 731) to a Hauptsatz,† though by inadvertence nothing has been so marked. The current of development is interrupted by

* Conformal except at winding points.

† The first paragraph of page 715, beginning with line 3, meets essentially the requirements of the references.

certain considerations of analysis situs and by the necessity of proving the well known theorem of Hurwitz on the number of roots of an analytic function $\phi(z) = \lim_{n=\infty} \phi(n, z)$ in a given region.

I have already paused unduly on the one place in the entire volume where the author falls below a uniformly high level of execution. Not a little trouble would be saved the reader if some such orientation as the following had been placed early in the chapter instead of being left to emerge at the end. The fundamental geometrical problem underlying the chapter is that of the conformal mapping of a *simply-connected* region of *interior* points upon the whole or a part of the smooth plane. The region may or may not have a frontier and may contain an infinite number of leaves, but the winding points, so far as they are included as interior points, are to be of finite order (cf. Study's presentation). At such points conformality of representation is, of course, renounced. Three cases arise in the mapping according as the region can be referred to (1) the entire plane, (2) the finite plane, and (3) the interior of the unit circle. It is, indeed, by the renunciation of all boundary considerations that the recent advance has been made by Osgood, Koebe, and Poincaré. When the conformal representation is effected for the simply connected interior of the Riemann surface properly constructed for an analytic function $w = f(z)$, the uniformization by pairs of one-valued analytic functions $z = \phi(t)$, $w = \psi(t)$ at once follows as a corollary, three cases being distinguished according as the domain of the variable t is one or another of the three regions given above.

In this uniformization work Osgood had before him the problem of interpreting the fat pages of Koebe and of getting at their kernel. The task is very far from an easy one, and he deserves our hearty thanks for having accomplished it in brief compass and with great penetration and clearness. The desirability of such a presentation was also perceived in another quarter, and shortly after the publication of Osgood's new edition there also appeared an independent but in some ways different treatment at the opening of the second Heft of Study's Vorlesungen über ausgewählte Gegenstände der Geometrie. The reader will find it most interesting to compare the two presentations. Study and Blaschke throughout the Heft employ Osgood's first edition for reference and, in particular, in their exposition of Koebe assume many of the results contained in its last chapter.

Koebe's Abbildungsatz (Osgood II 14, § 12) should be pointed out as an especially happy hit, for by means of it the difficult case in which there is no Green's function for the region can be controlled, the surface being then built upon the open finite plane (case 2 above). The greater definiteness imparted to this lemma by Osgood should also be remarked (cf. footnote, page 727). The other two cases (Nos. 1 and 3) can be handled with or without the lemma and are discharged by Osgood through older and less artificial methods. The volume closes with a proof of the very fundamental point—which, so far as I know, was neglected by Koebe—that to every analytic function there corresponds a Riemann's surface, not merely im Kleinen but im Grossen; and conversely, under certain restrictions, to every simply connected Riemann surface an analytic function.

II.

It remains yet to consider the characteristics of the work as a whole. In good measure we knew what to expect from Professor Osgood's pen; clearness, exactness, penetration, refined distinctions and proofs, discrimination between the vital and unessential. In such anticipations, we are not disappointed. The work fairly pulsates with these characteristics. They are so apparent that it will be unnecessary to dwell upon them. To these temperamental qualities there is added the fullness of knowledge which began but was by no means completed in the preparation of the section upon the theory of functions in the Encyklopädie der mathematischen Wissenschaften. The influence of this encyclopædic training is to be seen in the diversity and fertility of content of the present volume. To some extent they impede the swing of matter and style. But there is careful ordering and close connection of parts. Indeed, the different paragraphs and sections are so closely knit together as to necessitate an unusual amount of cross referencing. The reader is kept looking both backwards and forwards and must proceed warily. The book has the prime merit that it can not be read without careful thinking.

Few subjects in mathematics offer such opportunity for variety of treatment as the theory of functions. The selection and disposition of material must depend largely upon the author's aim and his view of the subject. In the present instance we find the anomaly of a work from the Cauchy-Riemann standpoint by a man of Weierstrassian feeling. From remarks in the introduction regarding the use of power series it is clear that the Weierstrassian development does not seem so fundamental because it is tied up to one form of representation. In the author's language "the theorems gain in clearness by striking off the incidental element which enters in by formulation in terms of power series." Of the far greater rapidity of the Riemann-Cauchy method there can be, I think, no question, provided the necessary prerequisites and knowledge can be assumed. Also there is a gain in steeping the beginner in that method which has proved itself the most powerful but which it is more difficult to master as a tool. Never before has the Cauchy-Riemann development been given with such completeness and precision. To a considerable extent preceding expositions have been on a basis of loose analysis. The revision of the calculus and fundamental principles of analysis in recent time necessitates a reworking of the Cauchy-Riemann development. It is easy to imagine the appeal which such a recasting would make to one endowed with a Weierstrassian love for strict and legal proof. Here Osgood has seen his opportunity and thrown himself into the work.

As has been already said, a thorough comprehension of the Cauchy-Riemann basis requires a considerable acquaintance with analysis. It is accordingly presupposed that the student has a well grounded knowledge of the scientific principles of the calculus, not such a knowledge as is derived from a formal first course so often found advisable for the American student to supplement deficiencies of drill in preliminary subjects, but such a grip as may be gained from an extended second course concerned largely with its fundamental concepts. This course should also be accompanied with a good course in mechanics. It is the author's belief that it is desirable to keep the students in mathematics and physics together as long as possible in the study of analysis, that thereby the students of mathematics, on the one hand, shall perceive its close physical relations and applications, while, on the other hand, the students of physics shall secure a sound working knowledge of analysis, which is with them unfortunately altogether too rare. To achieve this it is highly desirable to carry the physical student into the function theory and to unroll it in close relation to the subject of potential. Such, as I conceive it, is the author's "Glaubensbekenntniss," the platform beneath his text book; and it is a strong one. It explains also the formation of his Abschnitt III, devoted largely to physical applications and the potential theory. It seems not unlikely that the structure and disposition of the material may exert a strong influence on the teaching of the theory of functions in at least our own country.

Incidentally it may be pointed out that another cause which may have contributed in some measure to the formation of Abschnitt III is the author's fondness for conformal representation. Attention to this subject is a conspicuous feature of the book almost from first to last. I know of no place where it is so adequately and broadly handled.

The discussion just given leads us naturally to consider the question how far the work is adapted for introductory use. The answer to this query must depend largely upon *how* the book is used. It is not an Einleitung for private study. A more descriptive and appropriate term is the word Lehrbuch used in the title. The beginner may, I fear, find it difficult to get the perspective. His feeling might be akin to that of a sightseer left alone to take a first view in an enormous art gallery like the Louvre. Masterpieces of design and portraiture are about him on all sides, and in their multitude he feels lost. Furthermore, the beginner rarely has at the start the capacity for distinctions necessary for proper reading of this book. Large omissions will therefore be advisable for him in a first reading. Certain recommendations in this regard are made at the opening of the fifth chapter but other omissions should also be made, such as naturally occur in following a course of lectures. But for collateral use in a lecture course the book is invaluable, and it is an indispensable handbook for every teacher of the function theory.

If a digression is in place here, a plea might be made for a *brief* introductory course in the Weierstrassian theory. The student becomes familiar in calculus with the Taylor's series. He will be quick therefore to see the importance of concepts and results arising in connection with its further study and will then be ready for application of the ideas to other series whose terms are not powers or power elements. At the same time concreteness of development is never lost. Inasmuch as the Weierstrassian theory requires less preliminary study, it can be used to introduce the student into the theory of functions at an earlier stage. Pedagogically, therefore, it has its advantages. Yet it is unquestionalby a narrow development and hence its study should not be pushed by the beginner too far. To a far less degree than the Cauchy-Riemann method it involves the perpetuation and broadening of the analysis which is begun with the calculus. Hence it seems not unlikely that an introduction from the other standpoint may become increasingly important and prevalent.

Before leaving general considerations I should not fail to reemphasize one aim and achievement. It is, I think, the *crowning* excellence of the work to have culled out of the heart of the analytic function theory the most vital methods and results and to have given a systematic development of the whole upon a secure foundation of analysis. How great a progress has been made may be seen by comparing the work with some of its predecessors.

I turn now to more detailed considerations. The theorems of Abschnitt I may be divided roughly into two classes according to their elementary or their advanced character. Those of the former class relate mainly to fundamental concepts of the function theories and are particularly well done. The other group concerns the arithmetization of analysis situs and is more exacting in the penetration required. It is the latter to which the specialist will turn and which the author has treated con amore.

As typical of this second group take the familiar theorem that a closed Jordan curve divides the plane into two distinct regions. Intuitionally the theorem is declared obvious, but mathematical progress has shown that there are curves and curves of astounding complexity, even possessing properties contradicting intuition. In Osgood's treatise the student is not allowed to jog along whistling, happy in the belief that the concept curve is something we know all about or is an irreducible element of thought too perfect for analysis. Osgood nowhere tells us what he thinks should be called a curve (we regret the omission), but Jordan curves, simple (i. e., "einfache") curves, analytic curves, regular curves are carefully defined. The above theorem is not proved for any closed Jordan curve but is established by the Ames method for any closed curve both simple and regular. It is, indeed, the regular curve, composed of a finite number of pieces with continuously turning tangent, which he chooses as the fundamental contour in deriving the integral theorems of Cauchy and others. The statement of these theorems with definite and reliable contour limitations is satisfying. As a characteristic example of what I mean I may refer the reader to the formulation of the familiar theorem (page 333) connecting

$$\frac{1}{2i\pi}\int\frac{f'(z)}{f(z)}dz$$

with the number of roots and number of poles of $f(z)$ in a given region.

A second conspicuous instance of the arithmetization of geometrical theorems is to be found in the treatment of regional theorems; in particular, the theorem that the integral $\int(Pdx + Qdy)$ taken around a contour vanishes if at every interior point of the included region we have $\partial P/\partial y = \partial Q/\partial x$, proper continuity conditions being imposed on P and Q and their derivatives. This result is first obtained on the basis of "Naïve Anschauung" and with the help of Green's lemma for transforming double integrals $\int\int\frac{\partial P}{\partial y}\,dxdy$, etc., into contour integrals. A brief critique of the method (page 132) then follows, and finally a strict proof involving the decomposition (cf. § 9, 10 of chapter 5) of a region with regular contour into a finite number of regions of certain standard forms, termed by the author σ-regions. In this we have some of the author's own work. Even as the Weierstrass curve without a tangent is beyond intuition, so also the shapes of regions may transcend the power of visualization. Hence theorems involving regional consideration require careful arithmetization to obtain an irreproachable basis. Corresponding care is taken in the dissection of a region to obtain the familiar theorems of connectivity (pages 172–176), the loop cuts and cross cuts being here restricted in generality. Like the closed contour in the Jordan theorem, the loop cuts are also assumed to be simple and regular, and the cross cuts are also regular except possibly at their end points. Attentive reading and careful thought are required in these sections of the work. In the new edition there is a noteworthy simplification in the treatment by means of the σ-regions.

Certain features in the first mentioned class of more elementary theorems also deserve mention; for example, the very simple and natural proof after Dini of the fundamental theorem on implicit functions, and the remarkably fine section giving the necessary and sufficient condition for conformal representation, a topic which is only too often slurred over in unsatisfactory fashion. A protest may well be made against the author's use of the term "infinite derivative" (pages 22, 26) to signify one which is either positively infinite or negatively infinite. This would debar the application of the term at a vertical cusp if the curve of the function lies on both sides of the cuspidal tangent.

The analytic function is first introduced in the opening chapter of Abschnitt II, together with its inverse. Then follows a detailed study of the elementary functions and the corresponding conformal representations. The general theory begins with the succeeding chapter, which is devoted to the consideration of functions which are single-valued in a given domain. The structure of this chapter and, in particular, the ordering of the theorems will be of especial interest to the reader. The development is rapid and clear. Here, as elsewhere, the author avoids the ϵ-methods which are of so little interest to the physicist, and in their stead, without sacrifice of rigor, he makes his proofs dependent upon fundamental principles and well-ordered results. Note also the excellent and careful formulation of the contour integration theorems throughout the chapter. Thus, for example, in the very first of these we have the pivotal theorem that an analytic function of z is defined by $\int\phi(t)dt/(t - z)$. Here the contour of integration, though restricted to be simple and regular, is allowed to be open as well as closed, while $\phi(t)$ is any function, real or complex, which is merely supposed to be continuous along the curve. The theorem is also typical of the special attention to parametric integrals, a topic of importance largely neglected in other works on the theory of functions.

The point at infinity is introduced surprisingly late (page 322). But after its introduction the rôle of this "uneigentlicher Punkt" is most carefully observed in theorem and proof. One amusing consequence of its late introduction is to be seen on page 285, where a footnote of the previous edition has been incorporated into the text, thereby making a theorem refer to the point at infinity before its introduction.

Chapter 9, on analytic continuation, is noteworthy on account of the generality and delicacy of its considerations. Painlevé's theorem for continuation (page 434) has here the commanding position, the Weierstrassian method of continuation by means of overlapping circles being given a subordinate place. The chapter closes with a commendable paragraph on the principle of permanence of a functional equation. A nice application is made to the analytic continuation of the Γ-function when defined in the usual manner for the right half-plane by means of an integral, also to the continuation of elliptic functions. With regard to the Weierstrassian treatment of the theory of functions on the basis of power series, it may be said that the sections relating to it are necessarily scattered, so that a view of the method as an organized whole and in its full strength is not directly afforded. Also it does not fall within the scope of the volume to consider in any way a large group of investigations which in one form or another rest on the determination of an analytic function by the coefficients of a power series.

The term "element" of an analytic function is not employed in the narrower Weierstrassian sense of a convergent Taylor's series but is extended to apply to all expressions of form

$$\phi(z) = \sum_{k=m}^{+\infty} c_k(z - a)^{k/n},$$

where n is a positive integer and m any integer positive, negative, or zero. The smooth or winding circular domain of an element may therefore have a pole or a branch point of finite order for its center. The wisdom and even necessity of this broader use becomes luminous in the last chapter, where the inclusion of such elements is indispensable for the uniformization of an analytic function. The truth is, as the author so clearly realizes, that the nature of the element must be adapted to the problem in hand. For many purposes a meromorphic element is obviously more useful than a holomorphic one. As for the Taylor's series with its system of overlapping circles, it is at best adapted to considerations involving a plane with one point lacking, i. e., the point at infinity.

The first two chapters of Abschnitt III gives a succinct and rapid development of fundamental properties of simply as well as doublyperiodic functions. In the succeeding chapter the elementary functions are considered from an individualistic standpoint unusual for such a treatise. Though the chapter belongs more strictly to the function theory of the real variable, it is nevertheless a welcome interruption, since it is both profitable and stimulating to see old functional friends in new lights. Divers definitions of the functions for real values of the arguments are given. Some of these conduct to characteristic functional equations, for which, conversely, the general solutions are then sought. It will be noted that the first of Cauchy's four fundamental types of functional equations,

$$f(x + y) = f(x) + f(y),$$

is relegated to an exercise, a subordinate position which is altogether natural because of the mode of approach to it but which is totally out of relation to its fundamental importance. The chapter closes with a novel and interesting method applicable in the *real* domain to the derivation of standard expansions for sin x, ctn x, etc., into infinite series and products which make visible their zeros and poles respectively.

It remains yet to speak of the chapter on logarithmic potential. In this we find a development of the theory of logarithmic potential in a series of theorems closely parallel to those of the function theory but on an independent basis. As is so well known, in addition to the Weierstrassian and Cauchy-Riemann bases for the function theory there is also a third one, which rests on the potential theory and follows on separating potential functions into conjugate pairs. I know of no place where this is so completely and delightfully presented as in Osgood's Lehrbuch. It has for us the further interest of being one to which Bôcher has contributed. Many of the theorems, as for instance that for continuation of a harmonic function across a boundary (III 13, § 6), must interest the physicist. I may also call attention to the theorem (page 673) that for a simply connected region the curve along which a Green's function keeps a constant value is a simple and regular closed curve without corners.

The work is distinguished from most or all other works on the theory of functions by a good supply of illustrative exercises. There is also a fine index both of names and matter which contributes greatly to its usefulness.

The system of cross referencing is fatiguing. Beside chapter and paragraph numbers (both in same type) there is often a Hauptsatz or Satz number to be looked for under the paragraph, and sometimes a Zusatz under the Satz. Inasmuch as the number of cross references is extremely large, the method employed becomes a matter of some importance to the reader. If page references are deemed impracticable because of trouble with the proof and because of the necessity of change with every new edition, it may be suggested that a system preferable to the author's would be a division of the subject matter into short paragraphs not more than a page or two in length and numbered consecutively to the end, as is done in so many French works. The terms and definitions are well chosen and, with rare exceptions, well explained. So far as I can find, the terms Randpunkt for a region and singular point for a function are left undefined, being regarded as self-explanatory or sufficiently clear from the examples given. It is to be regretted that a term so distinctive as "meromorphic" and so universally used is nowhere introduced.

In conclusion, admiration must be expressed for the combination of a grasp "im Grossen" with one "im Kleinen," to apply descriptive terms which are employed by the author and which, if we mistake not, are of his own mint. The appearance of the second volume will be awaited with great interest. Analytic functions of two or more variables will doubtless be considered? What else? From the abundance of material on every hand we may confidently expect to have again a selection of great individuality and interest.

EDWARD B. VAN VLECK.

PARIS,
March 27, 1914.

NOTES.

IN response to the invitation of Brown University to participate in the celebration of its one hundred and fiftieth anniversary, the twenty-first summer meeting of the American Mathematical Society will be held at that university on Tuesday and Wednesday, September 8–9. Titles and abstracts of papers intended for presentation at the summer meeting should be in the hands of the Secretary by August 22.

THE closing (June) number of volume 15 of the *Annals of Mathematics* contains the following papers: "Some solutions