$$S_n{}^{(k)}, \quad nS_n{}^{(k-1)}, \quad n^2S_n{}^{(k-2)}, \quad \cdots, \quad n^{k-1}S_n{}^{(1)}$$

is in absolute value less than B for every value of n. Suppose further that the series

$$\sum_{i=1}^{\infty} |\Delta^k c_i|$$

converges uniformly in D and

$$\lim_{n=\infty} \Delta^{k-1}c_n = 0.$$

Then the series

$$a_1 c_1 + a_2 c_2 + a_3 c_3 + \cdots$$

converges uniformly in D.

For the case $k = 1$ both of these theorems are already known. This special case of Theorem III appears to have been first employed by Nielsen[*]; but Nielsen's statement of it is not entirely accurate, as Landau[†] has pointed out. The corresponding special case of Theorem IV is due to Cahen.[‡]

INDIANA UNIVERSITY,
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A TRANSLATION PRINCIPLE CONNECTING THE INVARIANT THEORY OF LINE CONGRUENCES WITH THAT OF PLANE n-LINES.

BY PROFESSOR O. E. GLENN.

(Read before the American Mathematical Society, September 9, 1913.)

A WELL known translation principle of Clebsch[§] enables us to construct a ternary invariant by a simple algebraical operation upon a binary invariant. In brief, if the line $u_x = 0$ intersects the curve

$$f = a_x{}^n = b_x{}^n = \cdots = 0$$

in the n points given by the binary form

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[*] *Annali di Matematica* (3), 15 (1908), pp. 275–282.

[†] *Sitzungsber. d. K. Bayer. Akad. d. Wiss., Phys.-Math. Classe,* 1909.

[‡] *Annales scientifiques de l'Ecole Normale Supérieure* (3), 11 (1894), p. 79.

[§] Clebsch, Vorlesungen über Geometrie, vol. 1, p. 276; and *Journal für Math.,* vol. 59 (1861).

$$g = a_\xi'^n = b_\xi'^n = \cdots = 0,$$

and if

$$I = \Sigma C(a'b')(a'c') \cdots$$

is any invariant of g, then

$$I(u) = \Sigma C(abu)(acu) \cdots$$

is a contravariant of f; and it represents the envelope of $u_x = 0$ when the latter cuts $f = 0$ in a range having the projective property $I = 0$.

Let us, for the moment, call $f = 0$, $g = 0$ the higher and lower locus, respectively, and $u_x = 0$ the joining locus. Then a translation principle is an algebraical process depending upon the joining locus, which converts an invariant of the lower locus into an invariant of the higher locus.

In this note we establish a translation principle in which the higher locus is a line congruence in three-space, the lower locus a plane n-line, and the joining locus is a plane.

§ 1.   *The Congruence $C(n; 1)$.*

The line determined by two planes

$$u_x \equiv x_1 - p_1 x_3 - q_1 x_4 = 0, \quad v_x \equiv x_2 - p_2 x_3 - q_2 x_4 = 0$$

was designated by Plücker* as his element of three-space. As coordinates of one of the ∞^4 elements we have the non-homogeneous system

$$(p_1, q_1, p_2, q_2, r) \qquad\qquad [r = (pq)].$$

A non-homogeneous form in these variables represents ∞^3 elements forming a line complex.   Consider two such forms of orders n and unity respectively,

$$f_n(p, q, r) = 0,$$

(1)
$$f_1(p, q, r) \equiv \sum_{i=1}^{2} (\alpha_i p_i + \beta_i q_i) + \gamma_1 r + \gamma_2 = 0;$$

the ∞^2 elements common to these two complexes form a line congruence which we may represent by $C(n; 1)$.

The intersection of an arbitrary plane $w_x = 0$ and the congruence $C(n; 1)$ is an n-line.

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* Plücker, System der Geometrie des Raumes u. s. w. (1846).

An analytical proof is furnished as follows: Impose the condition that $w_x = 0$ pass through the element (u_x, v_x). This is

$$(2) \qquad \begin{Vmatrix} w & w_2 & w_3 & w_4 \\ -1 & 0 & p_1 & q_1 \\ 0 & -1 & p_2 & q_2 \end{Vmatrix} = 0,$$

or

$$(3) \qquad p_1 w_1 + p_2 w_2 + w_3 = 0, \quad q_1 w_1 + q_2 w_2 + w_4 = 0.$$

From (3) we get

$$(4) \qquad r \equiv (pq) = \frac{1}{w_1}(w_4 p_2 - w_3 q_2),$$

and the substitution of this in f_1 eliminates r and reduces f_1 to a linear non-homogeneous form in p_i, q_i $(i = 1, 2)$, involving the parameters w.

We now solve $f_1(p, q)$ with (3), thus expressing p_2, q_1, q_2 each as a linear expression in p_1, say

$$(5) \qquad \begin{aligned} p_2 &= g(w)p_1 + h(w), \\ q_i &= g_i(w)p_1 + h_i(w) \qquad (i = 1, 2). \end{aligned}$$

Substitute these in $f_n(p, q)$ and we get a non-homogeneous equation or form $F(p_1)$ in p_1, with coefficients rational in w_i $(i = 1, 2, 3, 4)$, and of order n. The n roots $p_1^{(j)}$ $(j = 1, 2, \cdots, n)$ put in (5) in turn, give us just n sets

$$(p_i^{(j)}, q_i^{(j)}, r^{(j)}) \quad (i = 1, 2; j = 1, 2, \cdots, n).$$

These sets are the coordinates of n lines of intersection as stated.

Now if we project the n-line in which $w_x = 0$ cuts $C(n; 1)$ upon the $x_3 = 0$ plane, we get the n-line whose equation is

$$(6) \qquad f_{3n} \equiv \prod_{j=1}^{n}(p_2^{(j)}x_1 - p_1^{(j)}x_2 + r^{(j)}x_4) = 0.$$

In the translation principle to be established $C(n; 1)$ figures as the higher locus, $f_{3n} = 0$ as the lower locus, and $w_x = 0$ the joining locus.

§ 2. *Translation Principle.*

It is easy to show by combining (4), (5), (6) that the n lines of f_{3n} are concurrent. Some of the full invariants of

this n-line therefore vanish identically. Thus when $n = 3$ the only full invariant, the condition for a triple point, vanishes identically. Now, however, even when $n = 3$ or $n = 2$ we always have non-vanishing invariants of the binary form $F(p_1)$, and these, through (6), have interpretations as invariants of the lower locus f_{3n}. For instance, if the discriminant of $F(p_1)$ vanishes, then f_{3n} has a double line.

Every rational invariant I of $F(p_1)$ is a rational form in w_i ($i = 1, 2, 3, 4$). Such a form represents an algebraical surface whose equation is in planar coordinates. *This surface, $\varphi(w) = 0$, is the envelope of the plane $w_x = 0$ when the latter moves so as to cut the congruence $C(n; 1)$ in the n-line having the projective property $I = 0$.*

It is to be noted that q_1, q_2, p_2 each satisfy an equation analogous to $F(p_1) = 0$. But each equation may be obtained from $F(p_1) = 0$ by a homographic transformation of $F(p_1)$, as is readily shown from (1), (3), (4). The determinants of these transformations will be different from zero, save when (1) and (3) are inconsistent. Hence the surfaces obtained from the equations in q_i, p_2 coincide with $\varphi(w) = 0$. In other words the translation principle gives a unique result, $\varphi(w)$.

§ 3.   *Class of $C(n; 1)$, etc.*

A conspicuous case of I is the discriminant of $F(p_1)$. Applying the translation principle, we get a surface $\varphi(w) = 0$ which is the envelope of $w_x = 0$ when the latter moves so as to touch $C(n; 1)$ in a double line. In this case $\varphi(w) = 0$ may be called the class equation of the congruence $C(n; 1)$. The order of $\varphi(w)$ is the class of $C(n; 1)$.

For illustration we take the special congruence given by

$$f_2(p, q) \equiv (pq) + p_1q_1 + ap_2 = 0,$$
(7)
$$f_1(p, q) \equiv (pq) + 1 = 0.$$

By (3), (4) we have

$$w_4p_2 - w_3q_2 + w_1 = 0,$$

$$w_1p_1 + w_2p_2 + w_3 = 0,$$

$$w_1q_1 + w_2q_2 + w_4 = 0.$$

Hence

(8) $F(p_1) \equiv w_2w_4p_1{}^2 - (w_2{}^2 + aw_1w_3)p_1 - w_3(w_2 + aw_3) = 0,$

and the class equation of the congruence reduces to

$$(9) \quad \varphi(w) \equiv (w_2{}^2 + a w_1 w_3)^2 + 4 w_2 w_3 w_4 (w_2 + a w_3) = 0.$$

It is known that every linear complex for which $(pq) \neq 0$ can be reduced* by projective transformation to the canonical form $(pq) + 1 = 0$. Hence we may take as a general case the congruence given by the general $f_n(p, q)$ and the special f_1 of (7).

It follows that $F(p_1)$ in general has coefficients which are homogeneous expressions of order $2n$ in w_i $(i = 1, 2, 3, 4)$. Hence since the discriminant of $F(p_1)$ is of degree $2(n - 1)$, *the class of the general congruence $C(n; 1)$ is $m = 4n(n - 1)$.*

In the special case above the class has been reduced by dividing out powers of w_i.

§ 4.   *Generalization to Congruence $C(n; m)$.*

Instead of the non-homogeneous system of Plücker (p_i, q_i, r) we may use the homogeneous coordinates of Grassmann, viz.,

$$(q_{12}, q_{13}, q_{14}, q_{23}, q_{42}, q_{34}),$$

where

$$q_{12} q_{34} + q_{13} q_{42} + q_{14} q_{23} = 0.$$

Then a congruence $C(n; m)$ is given by the two homogeneous equations

$$(10) \quad\quad f_n(q_{ik}) = 0, \quad g_m(q_{ik}) = 0.$$

To obtain the lines in which an arbitrary $w_x = 0$ intersects $C(n; m)$ let u_x be an arbitrary plane cutting w_x in an element of the complex f_n. Then

$$q_{ik} = u_i w_k - u_k w_i,$$

and the planar equation of the lines of the complex which are contained in $w_x = 0$ is

$$(11) \quad\quad f_n(u_i w_k - u_k w_i) = 0,$$

where u_j are the variables. Likewise $w_x = 0$ intersects the other complex in the lines

$$(12) \quad\quad g_m(u_i w_k - u_k w_i) = 0.$$

The lines (11) envelope a curve of class n† and (12) a curve of

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* Lie, Die Berührungstransformationen, p. 223.
† Lie, ibid., p. 286.

class m.   Hence $w_x = 0$ *cuts $C(n; m)$ in an nm-line*, by Bezout's theorem.*

If we project the curves (11), (12) upon the x_3 plane, we may obtain the equation of the projected mn-line by interchanging point and line coordinates in Clebsch's proof of Bezout's theorem (see Vorlesungen über Geometrie, page 282).   Every full invariant of this mn-line gives by our translation principle an equation of condition among the coordinates w_i.

An alternative method of procedure is to use equations (1), (3), replacing f_1 by g_m in (1).   Rational elimination processes give a form in each variable p_1, q_1, p_2, q_2 with coefficients rational in w_i.   Of these forms that in p_2 is the transformed of the one in p_1, say of $F_1(p_1)$, by a homographic transformation, and that in q_2 is likewise the transformed of the one in q_1, viz. $\varphi_1(q_1)$.   But φ_1 is not transformable into F_1.   As an invariant of the mn-line of intersection we may then select a simultaneous invariant of the binary forms $F_1(p_1)$, $\varphi_1(q_1)$, and by translation this invariant goes into an equation of condition in the variables w_i, representing a contravariant surface.

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SOME MATHEMATICAL BOOKLET SERIES.

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ENGLISH and French mathematical literature is entirely lacking in such admirable booklets dealing with elementary topics, as those which have wide circulation in Germany and Italy.†   I refer to the Mathematische Bibliothek of the

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* Bezout, Theorié générale des Equations algébriques (1779).

† It may be suggested that the volumes on Elimination by Laurent and on Geometrography by Lemoine, of the excellent " Scientia " series (Gauthier-Villars, Paris) are elementary, but these are only two of a dozen volumes by Appell, Gibbs, Hadamard, Poincaré, etc., which certainly may not be classed in this way.   And even these two brochures are more