shows that my statement was inaccurate, and he suggested also the hypotheses of Professor Moore which are given above.

G. A. BLISS.

REID'S THEORY OF ALGEBRAIC NUMBERS.

The Elements of the Theory of Algebraic Numbers. By LEGH WILBER REID, with an Introduction by DAVID HILBERT. New York, The Macmillan Company, 1910. xix + 454 pp.

PROFESSOR Reid's book is the result of an attempt to present the theory of algebraic numbers through a somewhat detailed study of the numbers belonging to special quadratic realms. The first four chapters, constituting one third of the whole book, are devoted to the theory of rational integers with special emphasis upon those properties of these numbers "which find their analogues in the general theory." There can be no question but that this procedure helps the reader to appreciate the character of the generalization made through the introduction of the ideal numbers, but it is difficult to see why it is necessary to use so much space for the introduction of material easily available elsewhere, when such a course makes it necessary to omit some of the more important parts of the theory for which the book was written. The presentation of this material is good, though rather lengthy.

In the proof of the unique factorization law for rational integers much emphasis is laid upon the sequence of three well-known theorems. The first of these affirms the validity of the euclidean algorism for finding the highest common factor of two integers; the second relates to the solution in integers of the equation

$$ax + by = 1,$$

where a and b are relatively prime; while the third asserts that when the product of two integers is divisible by a prime at least one of the integers is divisible by the prime. These three theorems the author refers to as Theorems A, B, and C, respectively.

Chapters V, VI, and VII are devoted to the discussion of the integers in the realms $K(i)$, $K(\sqrt{2})$, and $K(\sqrt{-3})$. The basis, the discriminant, the units of the realm are discussed separately for each one of these realms and without reference to the general quadratic realm, though, as the author points out, many of the proofs apply directly to the general case. For each of the three realms the unique factorization law is proven exactly as in the realm of rational integers, i. e., Theorems A, B, and C with proper modifications are shown to hold. The constant repetition certainly serves to fix the main points in the reader's mind, but it grows somewhat monotonous.

In Chapter VIII the whole process is begun over again for the special realm $K(\sqrt{-5})$, and an effort is apparently made to bring the reader with as great a shock as possible to the realization of the fact that in the special realm $K(\sqrt{-5})$ the euclidean algorism can not be carried out and, consequently, the unique factorization law can not be proven by methods used for the realms previously discussed. Be that as it may, the statement of Theorem A is given for the fifth time and this time for a realm for which it is not true. This method of presentation is not without its advantages, but the book would have been less open to criticism if the author had not gone to the trouble of dignifying an untrue statement by the name of a theorem, solely for the purpose of disproving it. Indeed, since it has been known for a long time that Theorem A holds by limitation only, it is doubtful if it deserves the prominence given it by the author in a book designed to be an introduction to the general theory.

Having shown that the unique factorization law can not be proven for $K(\sqrt{-5})$ by the method employed in demonstrating it for the realms previously discussed, the author introduces a few simple examples to show that it is not true, not only for this realm, but for some other realms as well, and proceeds at once to the introduction of ideals defined for the special quadratic realm $K(\sqrt{-5})$. Several definitions, such as the definitions for an equality of ideals, principal ideals, multiplication of ideals, unit and prime ideals, are given only to be repeated in almost identical words in a later chapter. However much we may be inclined to criticise the repetition that is to be found in these chapters on special quadratic realms, we must recognize the fact that they contain a mass of concrete illustrations which will be of great service to the beginner.

The valuable part of the book so far as American readers are concerned is to be found in the last six chapters. In these chapters the author discusses General theorems concerning algebraic numbers, the General quadratic realm, Ideals of a quadratic realm, Congruences whose moduli are ideal, Units of the general quadratic realm, and Ideal classes of a quadratic realm. The presentation is clear and is illustrated at nearly every point by numerous well-chosen numerical examples that will be a most welcome help to the beginner. Great pains are taken to point out analogies existing between the quadratic ideals and the algebraic integers.

The general excellence of this part of the book is marred by an unfortunate error on page 316 in what purports to be a proof of the theorem that the ideals of any realm for which the unique factorization holds are all principal ideals. In place of the erroneous statement, already noticed in a review by Professor Dickson,* the author has substituted the following to replace lines 5–16: "The converse of this, that, whenever the unique factorization theorem in its usual form holds for the integers of the realm, the ideals of the realm are all principal ideals, is true, but a further development of the theory of ideals is necessary for its proof. If, however, Theorem B holds for any realm, it is evident that not only does the unique factorization theorem hold but if

$$\mathfrak{a} = (\alpha_1, \alpha_2, \cdots, \alpha_r)$$

be any ideal of the realm and δ the greatest common divisor of $\alpha_1, \alpha_2, \cdots, \alpha_r$, then, as in $K(i)$, there exist integers

$$\xi_1, \xi_2, \cdots, \xi_r,$$

such that

$$\alpha_1\xi_1 + \alpha_2\xi_2 + \cdots + \alpha_r\xi_r = \delta.$$

Hence, we have

$$\mathfrak{a} = (\alpha_1, \alpha_2, \cdots, \alpha_r) = (\alpha_1, \alpha_2, \cdots, \alpha_r, \delta) = \delta,$$

a principal ideal."

This correction has been embodied in a two page list of "Additional Errata" printed for distribution since the first copies of the book were sent out.

It may seem like mere quibbling to say that the author fails to place in the clearest possible light the exact meaning of the

* *Science*, n. s., vol. 33, p. 188.

phrase "the restoration of the unique factorization law." On page 254 we read "When this factorization has been performed we find that every integer of $K(\sqrt{-5})$ can be represented in one and only one way as the product of prime ideal numbers." After making this direct statement, it is scarcely fair to the beginner for the author to relegate to a footnote the important statement: "We speak of the factorization of an integer α into its ideal factors, meaning thereby always the factorization of the principal ideal defined by α."

The chief defect of the book appears to be an over-elaboration of the material which is intended to be introductory to the main purpose. This elaboration of elementary material so increases the bulk of the book that any mention of the applications of the theory of quadratic ideals, or any development of the theory for higher realms is impossible. Some account of the relation between quadratic ideals and quadratic forms, and a brief account of the geometric interpretation of ideals would have added greatly to the value of the book. Moreover, to justify the title, the author was in duty bound to outline at least the simpler parts of the theory for the general case.

On the other hand, the latter half of the book is the easiest available approach for our American students to the elements of one of the most beautiful of modern mathematical creations. Furthermore, the author has put in a very clear light the historical sequence of the ideas which led to the development of the theory.

A few minor errors in proof reading, not given in the list of "Additional Errata" mentioned above, have been discovered by the reviewer. The book is well printed and contains an excellent index. A particularly valuable feature is the large number of numerical examples, many of them worked out, which are scattered throughout the text.

It is perhaps in order to mention an earlier review of this book which appeared in *Science* for February 3, 1911. In this review a distinguished mathematician, who has not even given the author's name correctly, has criticized the book severely in point of both accuracy and form of presentation. The reviewer has pointed out with clearness and force the weak points of the book, but has made some statements which, coming as they do from one of the ablest mathematicians of this country, ought not to be allowed to pass unnoticed.

For example, the reviewer commenting upon the author's method of showing that Theorem A does not hold for the realm $K(\sqrt{-5})$, says "after stating Theorem A and devoting fifteen lines to its proof, the author informs us that the theorem therefore fails." The failure of Theorem A, or its equivalent, is a fact of prime importance in bringing to light the simplest example of an "incomplete holoidal realm," the recognition of which was the starting point for the introduction of the whole theory of ideals. Moreover, the context makes it perfectly clear, even before going to the proof of the fact, that the theorem fails for $K(\sqrt{-5})$. Concerning the method employed differences may well exist, as I have already intimated, but the quotation given above puts the author in a wholly erroneous light.

In speaking of topics omitted from the book the same reviewer includes the class number. If he had gone over the table of contents carefully, he would have found that more than half of the last chapter, containing twenty-five pages, is given up to the class number.

Again, the reviewer, deploring the omission of references, says: "But to give hundreds of references to a certain report (excellent though it may be) and to completely ignore the literature and not even mention the names of the discoverers of the theorems, is against all scientific traditions." There are in the book one hundred and fifty-eight foot-note references to authors. Of these thirty-eight, not "hundreds," refer to Hilbert's Report. Frequent mention is made of the work of Fermat, Gauss, Jacobi, Dirichlet, Dedekind, Kummer, Kronecker, Minkowski, and the references are usually exact. We find, indeed, no reference to the books of Hensel or of König, and it is perhaps unfortunate that Sommer's excellent "Vorlesungen über Zahlentheorie," to which several references are made, is omitted from the half page list of books and reports given in the preface. But taking it all in all, the references are about as numerous as one should expect in a strictly elementary book.

To the present writer it seems that criticism based in part, at least, upon careless and inaccurate statements, and freely interspersed with exclamation points, is, to use the former reviewer's own phrase, "against all scientific traditions."

E. B. SKINNER.