THE TWENTIETH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE twentieth summer meeting and seventh colloquium of the Society were held at the University of Wisconsin during the week September 8–13, 1913. The attendance, which exceeded that of any previous summer meeting, included the following fifty-seven members:

Professor R. C. Archibald, Professor R. P. Baker, Professor G. N. Bauer, Professor G. D. Birkhoff, Professor H. F. Blichfeldt, Professor G. A. Bliss, Professor Oskar Bolza, Dr. H. T. Burgess, Professor W. H. Bussey, Professor R. D. Carmichael, Dr. G. R. Clements, Dr. A. R. Crathorne, Professor D. R. Curtiss, Professor S. C. Davisson, Professor L. E. Dickson, Professor L. W. Dowling, Professor Arnold Dresden, Dr. F. T. H'Doubler, Professor E. R. Hedrick, Dr. T. H. Hildebrandt, Dr. Dunham Jackson, Dr. A. J. Kempner, Professor Kurt Laves, Professor G. H. Ling, Professor A. C. Lunn, Professor H. W. March, Professor Max Mason, Mr. J. S. Mikesh, Professor C. N. Moore, Professor E. H. Moore, Professor W. F. Osgood, Professor B. L. Remick, Professor R. G. D. Richardson, Professor W. J. Risley, Professor W. H. Roever, Dr. J. E. Rowe, Miss I. M. Schottenfels, Mr. A. R. Schweitzer, Professor J. B. Shaw, Dr. H. M. Sheffer, Mr. T. M. Simpson, Professor E. B. Skinner, Professor H. E. Slaught, Professor C. S. Slichter, Professor E. R. Smith, Dr. E. B. Stouffer, Professor E. J. Townsend, Professor A. L. Underhill, Dr. S. E. Urner, Professor E. B. Van Vleck, Professor E. J. Wilczynski, Professor F. B. Wiley, Professor R. E. Wilson, Professor R. M. Winger, Professor H. C. Wolff, Professor B. F. Yanney, Professor Alexander Ziwet.

The four sessions of the summer meeting proper occupied the first two days of the week. The first session opened with an address of welcome by Professor C. S. Slichter on behalf of the University of Wisconsin and the local members of the Society. At this session, and also at the final session, the President of the Society, Professor E. B. Van Vleck, occupied the chair. Professor Bolza presided at the second, and Professor Osgood at the third session. The Council announced the election of the following persons to membership in the Society: Mr. W. E. Anderson, University of Minnesota; Professor W. O. Beal, Illinois College; Dr. C. A. Fischer, Columbia University; Professor A. E. Landry, Catholic University of America; Lieutenant Salih Mourad, Ottoman Navy; Miss E. A. Weeks, Mount Holyoke College. Thirteen applications for membership in the Society were received.

The Secretary reported that Columbia University had provided a separate office for the Society and that under the new plan of administration the work of the Secretary, Treasurer, Librarian, Committee of Publication, and the sales department would now be handled in this one office with the assistance of a clerk.

Professor L. E. Dickson was elected Editor-in-Chief of the *Transactions*. It was decided to hold the summer meeting of 1915 at San Francisco, in connection with the Panama Exposition.

The Register of the Society will hereafter be issued at intervals of two or three years. In the intervening years only a list of officers and members will be published.

The arrangements made by the local committee for the comfort and entertainment of the members throughout the week were most hospitable. No place in the middle west could be more ideal for such a series of meetings than Madison. The spacious lecture halls, the beautiful campus occupying an elevated position overlooking the capitol building and the adjacent lakes, Mendota and Monona, the commodious University Club, used as headquarters, and the hospitality of President Van Vleck and other members of the faculty who opened their homes for the entertainment of the members, —these and many other items contributed to the success of the farthest west summer meeting, except that at St. Louis (1904), and the only western colloquium of the Society.

On Monday evening, Professor Van Vleck entertained at dinner the members of the Council and the colloquium lecturers. On Wednesday afternoon the committee provided a two-hours' special excursion on Lake Mendota, ending at the Golf Club House in time for the dinner, at which fifty-five persons gathered. President Van Vleck, acting as toastmaster, called upon Professors Osgood, Bolza, Moore, Blichfeldt, Dickson, and Dr. Jackson for informal talks, and Professor Slaught read a telegram to be sent to the Secretary, expressing appreciation of his services to the Society and great regret at his enforced absence. At the close of the dinner Professor Ziwet voiced the unanimous sentiment in expressing thanks to the University and to the committee on arrangements for their generous hospitality. The dinner was followed by a moonlight ride on the lake back to the University Club. On Thursday the members were conducted by Professor Skinner about the campus and buildings of the University; and on Friday an automobile ride, provided by the mathematical faculty of the University and their friends, gave the members a fine opportunity to see the immediate surroundings of Madison. This ended in a most enjoyable buffet dinner at the home of President Van Vleck.

An account of the colloquium will appear later in the BULLETIN.

The following papers were presented at the four sessions of the summer meeting.

(1) Dr. E. B. LYTLE: "Note on iterable fields of integration."

(2) Professor W. H. BUSSEY: "The tactical problem of Steiner."

(3) Dr. JOSEPHINE E. BURNS: "The abstract definitions of the groups of degree eight."

(4) Professor WILLIAM MARSHALL: "The functions of the parabolic cylinder."

(5) Professor L. C. KARPINSKI: "The algorism of John Killingworth."

(6) Professor R. D. CARMICHAEL: "On series of iterated linear fractional functions."

(7) Professor R. D. CARMICHAEL: "Some theorems on the convergence of series."

(8) Mr. T. E. MASON: "The character of the solutions of certain functional equations."

(9) President E. B. VAN VLECK and Dr. F. T. H'DOUBLER: "On certain functional equations."

(10) Professor OSKAR BOLZA: "On the so-called 'abnormal' case of Lagrange's problem in the calculus of variations."

(11) Professors E. R. HEDRICK and W. D. A. WESTFALL: "An existence theorem for implicit functions."

(12) Professor R. G. D. RICHARDSON: "A solution of the Rayleigh minimum problem in the theory of sound."

(13) Professor G. C. EVANS: "The Cauchy problem for integro-differential equations."

(14) Dr. DUNHAM JACKSON: "A formula of trigonometric interpolation."

(15) Mr. J. W. ALEXANDER II: "Proof of the invariance of certain constants of analysis situs."

(16) Dr. J. E. ROWE: "On Fermat's theorem and related theorems (first paper)."

(17) Dr. J. E. ROWE: "On Fermat's theorem and related theorems (second paper)."

(18) Professor MAXIME BÔCHER: "The infinite regions of various geometries."

(19) Professor W. F. OSGOOD: "On functions of several variables which are meromorphic or analytic at infinity."

(20) Professor W. F. OSGOOD: "Note on line integrals on an algebraic surface $f(x, y, z) = 0$."

(21) Professor E. H. MOORE: "On a class of continuous functional operations associated with the class of continuous functions on a finite linear interval" (preliminary communication).

(22) Mr. A. R. SCHWEITZER: "On a general category of definitions of betweenness."

(23) Mr. A. R. SCHWEITZER: "The theory of linear vectors in Grassmann's extensive algebra."

(24) Mr. A. R. SCHWEITZER: "Remarks on functional equations."

(25) Mr. A. R. SCHWEITZER: "The general logical significance of uniformity of convergence of series."

(26) Professor EDWARD KASNER: "On the ratio of the arc to the chord for analytic curves."

(27) Dr. E. L. DODD: "The arithmetic mean as approximately the most probable value a posteriori under the Gaussian law."

(28) Professor E. J. WILCZYNSKI: "On the surfaces whose directrix curves are indeterminate."

(29) Professor J. B. SHAW: "On the transverse of a linear vector operator of n dimensions."

(30) Professor FLORIAN CAJORI: "Zeno's arguments on motion."

(31) Professor O. E. GLENN: "Note on a translation principle connecting the invariant theory of line congruences with that of plane n-lines."

(32) Professor F. R. SHARPE: "Conics through inflections of self-projective quartics."

(33) Professor F. R. SHARPE and Dr. C. F. CRAIG: "Plane curves with consecutive double points."

(34) Dr. MILDRED L. SANDERSON: "A method of constructing binary modular covariants."

(35) Dr. H. M. SHEFFER: "Superpostulates: introduction to the science of deductive systems."

(36) Dr. H. M. SHEFFER: "A set of six independent postulates for Boolean algebras."

(37) Professor R. M. WINGER: "Self-projective rational sextics."

(38) Professor R. M. WINGER: "Self-projective rational septimics" (preliminary report).

(39) Professor M. FRÉCHET: "Sur la notion de différentielle d'une fonction de ligne."

(40) Professor KURT LAVES: "A new theorem concerning the motion of two satellites of finite masses circulating in nearly commensurable motions of type $\frac{1}{2}$ about a central and homogeneous body of ellipsoidal shape."

(41) Professor H. F. BLICHFELDT: "On the order of linear homogeneous groups (fifth paper)."

(42) Professor T. R. RUNNING: "Graphical solutions of differential equations between two variables."

(43) Professor R. P. BAKER: "The genus of a group."

(44) Professor R. P. BAKER: "The topological configurations occurring in finite geometries."

(45) Professor R. D. CARMICHAEL: "On Fermat's theorem and certain related theorems."

(46) Professor H.W. MARCH: "Integral and series representations of an arbitrary function in terms of spherical harmonics."

Dr. Burns's paper was communicated to the Society through Professor Miller; Professor Fréchet's through Professor Dickson. Mr. Mason was introduced by Professor Carmichael; Dr. Sanderson by Professor Dickson. The paper of Professor Bôcher was read by Professor Osgood. The papers of Dr. Lytle, Miss Burns, Mr. Alexander, Dr. Dodd, Professors Marshall, Karpinski, Evans, Cajori, Glenn, Sharpe, Kasner, Running and Baker, the third paper of Professor Carmichael, and all the papers of Mr. Schweitzer except the first were read by title. Abstracts of the papers follow, with numbers corresponding to the titles in the above list.

1. In a former paper in the *Transactions*, volume 11 (1910), page 25, Dr. Lytle showed that a property, which he defined

as iterability of the field of integration, was a sufficient condition for a certain integral inequality. In the present note he shows that iterability is also necessary, and thus finds a necessary and sufficient condition that all limited functions simultaneously satisfy this inequality.

2. In this BULLETIN, volume 16, pages 19–22, Professor Bussey proved the existence of an arrangement in triads, tetrads, etc., for every number n of the form $2^j - 1$. In the present paper he proves the same thing in a more simple way and gives a method by which the triads, tetrads, etc., can easily be written down in any particular case.

3. Dr. Burns points out that the symbolic definitions of the abstract groups which may be represented as substitution groups on a small number of letters are effective in displaying the fundamental abstract properties of that important class of groups. The problem of finding the abstract definitions of the substitution groups of degree eight necessarily involves the detailed study of a large number of distinct groups, but the work may be materially simplified by considering several larger categories, each of which embraces several of the desired definitions. Such categories are represented by all groups which may be generated by two operators satisfying the following fundamental conditions:

1. $s_1^4 = s_2^4, \quad (s_1s_2)^2 = 1, \quad s_1^{-1}s_2s_1 = s_2^{\beta}s_1^{\alpha};$
2. $s_1^4 = s_2^4, \quad (s_1s_2)^2 = 1, \quad s_1^2s_2^2 = s_2^2s_1^2;$
3. $s_1^4 = s_2^4, \quad (s_1s_2)^3 = 1, \quad (s_1^2s_2)^2 = 1;$
4. $s_1^6 = s_2^6, \quad (s_1s_2)^2 = 1,$ s_1^3 and s_2^3 invariant;
5. $s_1^6 = s_2^6, \quad (s_1s_2)^2 = 1, \quad s_1^2s_2 = s_2s_1^2;$
6. $s_1^8 = s_2^4, \quad (s_1s_2)^2 = 1, \quad s_1^2s_2 = s_2s_1^2.$

In each case the equations given define a small category of abstract groups of finite order, many of which may be represented as substitution groups on eight or a smaller number of letters.

4. It is the object of Professor Marshall's paper to place the so-called functions of the parabolic cylinder on the same footing, as far as the possibility of numerical calculation is

concerned, as Bessel's functions, for example. To this end power series valid in the entire plane are deduced, which series contain only a single parameter, namely the ratio of the two constants κ and λ which appear in the defining equation

$$\frac{d^2U}{du^2} + (\kappa^2u^2 + \lambda)U = 0.$$

Then semi-convergent expansions suitable for computation when the argument is large are developed, and the connection between these and the convergent series established. Finally, formulas for the roots of these functions are deduced. The paper is to appear in the *Archiv der Mathematik und Physik.*

5. Prominent among the English mathematicians and astronomers of the fifteenth century was John Killingworth of Merton College, Oxford. Professor Karpinski presents a study of Killingworth's algorism, which is preserved in manuscript Ee. III. 61, Cambridge University Library. This work, which explains our present system of arithmetic, was written in 1444 A. D. From the standpoint of the history of method and the development of science in England this is an important document. The study shows that the technique of the arithmetical operations as taught in Oxford had improved but little in the centuries between Adelard of Bath (c. 1125) and Killingworth. A comparison with the writings of contemporaries, such as Peurbach of Vienna and de Beldamandi of Italy, indicates that much greater progress in the art of arithmetic had been made on the continent. Doubtless the faulty instruction in arithmetic contributed to the general decline of mathematical science in England, which continued until the time of Newton.

This paper will appear in the *English Historical Review.*

6. Among the most important series known to mathematical analysis are the two classes of power series (ascending and descending) and the factorial series. Closely associated with the latter are the so-called binomial coefficient series. In his first paper Professor Carmichael introduces a common generalization of descending power series and factorial series and also of ascending power series and binomial coefficient series. These are the series of iterated linear fractional functions

$$\alpha_0 + \frac{\alpha_1}{x} + \sum_{n=1}^{\infty} \frac{\alpha_{n+1}}{xS_1(x)S_2(x) \cdots S_n(x)},$$

$$\alpha_0 + \sum_{n=1}^{\infty} \alpha_n S_1(x) S_2(x) \cdots S_n(x),$$

where

$$S_1(x) = \frac{ax+b}{cx+d}, \quad ad-bc \neq 0; \quad S_k(x) = \frac{aS_{k-1}(x)+b}{cS_{k-1}(x)+d}, \quad k > 1,$$

a, b, c, d being constants. For the case where the substitution $x' = S_1(x)$ has two double points and the multiplier of the substitution is in absolute value equal to unity, the theory of both series is altogether exceptional; and consequently this case is not treated in detail. The general case subdivides into several special cases in each of which the general theory is essentially different from that in all the others. In each of these cases the fundamental elements of a general theory are developed. The nature of the region of convergence, of absolute convergence, of conditional convergence, and of uniform convergence is determined in every case; precise formulas also are obtained for the magnitude of these regions in terms of the coefficients of the series. It should be noted that the general results are in some respects in marked contrast to the simpler ones for the special cases which have been investigated heretofore.

7. No general criteria are known for the study of the convergence of a series of complex terms when the series does not converge absolutely. The object of Professor Carmichael's second paper is to generalize one of the few special methods which are already available for such study. Typical of the theorems obtained is the following: Let $a_1, a_2, a_3, \cdots$ be any infinite sequence of numbers such that each of the limits

$$\lim_{n=\infty} S_n^{(k)}, \quad \lim_{n=\infty} nS_n^{(k-1)}, \quad \lim_{n=\infty} n^2 S_n^{(k-2)}, \quad \cdots, \quad \lim_{n=\infty} n^{k-1} S_n^{(1)}$$

exists and is finite, where

$$S_n^{(1)} = \sum_{i=1}^{n} a_i; \quad S_n^{(t)} = \sum_{i=1}^{n} S_i^{(t-1)}, \quad (t > 1).$$

Let $c_1, c_2, c_3, \cdots$ be an infinite sequence of numbers such that $|\Delta^k c_1| + |\Delta^k c_2| + |\Delta^k c_3| + \cdots$ is convergent, where Δ is the difference symbol of the difference calculus. Then the series $a_1c_1 + a_2c_2 + a_3c_3 + \cdots$ is convergent (but not necessarily absolutely convergent).

8. Mr. Mason studies the solutions of the equations

$$\varphi(x+1)=\frac{A(x)\varphi(x)+B(x)}{C(x)\varphi(x)+D(x)} \quad \text{and} \quad \varphi(qx)=\frac{A(x)\varphi(x)+B(x)}{C(x)\varphi(x)+D(x)},$$

where $A(x)$, $B(x)$, $C(x)$, $D(x)$ are rational, with respect to their rational, algebraic non-rational, algebraically transcendental, and transcendentally transcendental character. A partial treatment of these problems has been given by Tietze and Stridsberg. In the present paper methods are developed for finding all possible rational and algebraic non-rational solutions. For the case where $C(x) \not\equiv 0$ and there are no algebraic solutions, if we exclude a certain well-defined and quite restricted class of equations of exceptional character, then a necessary and sufficient condition that an equation of either of the types above have an algebraically transcendental solution is that a certain third order difference equation have a rational solution. The author shows how to find all the possible rational solutions of this equation. These algebraically transcendental functions also satisfy a certain Riccati differential equation with rational coefficients. These functions are asymptotic to the formal solutions of the equation in power series in $1/x$ or in $1/\sqrt{x}$. Equations are given which have no algebraic solutions but which do have algebraically transcendental solutions. The author finds a necessary and sufficient condition that an equation of the forms considered has only transcendentally transcendental solutions. A list of comprehensive classes of equations which have only transcendentally transcendental solutions is given. In the case when $A(x) \equiv D(x) \equiv 0$, there is a simple necessary and sufficient condition for algebraic solutions; if there are no algebraic solutions, then all the solutions are transcendentally transcendental.

9. The paper of Professor Van Vleck and Dr. H'Doubler is devoted to a treatment of the functional equations

$$\phi(x+y)\phi(x-y) = \phi^2(x)\phi^2(y) - \psi^2(x)\psi^2(y),$$
$$\psi(x+y)\psi(x-y) = \psi^2(x)\psi^2(y) - \phi^2(x)\phi^2(y).$$

From these equations the properties of the Jacobi ϑ-functions can be rapidly unrolled. The only real solutions not trivial are, except for certain constant factors, pairs of ϑ-functions, or degenerate cases of the latter, such as the sine and cosine. In the complex plane there exist solutions of the functional equations which are non-analytic.

10. In his classical paper on Lagrange's multiplier rule, A. Mayer met with a certain exceptional case, which has since received the name of the "abnormal case of Lagrange's problem" (anomaler Fall, Hahn). In the present paper, Professor Bolza gives a detailed discussion of this abnormal case for the following generalized Lagrange-Mayer problem: To minimize the expression

$$U = \int_{t_0}^{t_1} f(y_1, \cdots, y_n, y_1', \cdots, y_n')dt + G(y_{10}, \cdots, y_{n0}; y_{11}, \cdots, y_{n1})$$

with respect to the totality of curves

$$y_i = y_i(t) \quad (t_0 \leqq t \leqq t_1; \quad i = 1, 2, \cdots, n),$$

which satisfy p differential equations

$$\varphi_\alpha(y_1, \cdots, y_n, y_1', \cdots, y_n') = 0 \quad (\alpha = 1, 2, \cdots, p),$$

q finite equations

$$\psi_\beta(y_1, \cdots, y_n) = 0 \quad (\beta = 1, 2, \cdots, q),$$

and r initial conditions

$$\chi_\gamma(y_{10}, \cdots, y_{n0}; y_{11}, \cdots, y_{n1}) = 0 \quad (\gamma = 1, 2, \cdots, r),$$

$(y_{10}, \cdots, y_{n0})$ and $(y_{11}, \cdots, y_{n1})$ denoting the coordinates of the two end points of an admissible curve.

11. In this paper Professors Hedrick and Westfall prove a series of theorems on the existence of the solutions of implicit equations without any assumption concerning the existence of any derivative. The theorems cover the existence of solutions of single equations for one unknown, and of systems of equations; and they yield the classical theorems in those cases in which the derivatives exist. The fundamental assumption in the case of one equation $f(x, y) = c$ is that the difference quotient $[f(x, y) - f(x_0, y_0)]/[y - y_0]$ be definite in a region about the fixed point (x_0, y_0).

12. What is the form of that plane region of unit area which furnishes a minimum value to the particular characteristic parameter value λ of the differential equation

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}+\lambda z = 0$$

which corresponds to a solution vanishing on the boundary? In other words, what is the shape of that drum head of prescribed area which emits the lowest fundamental tone? By means of a method similar to that used in simpler problems of the calculus of variations, Professor Richardson shows to be correct the surmise of Lord Rayleigh that the region must be circular. The desired solution of the differential equation is the solution of the following minimum problem: Of all functions $z(x, y)$ which vanish on the boundary C of a unit region R in the plane and which satisfy the condition

$$\int_R z^2 dR = 1,$$

which function z and which region R give to the integral

$$\int_R \left\{ \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 \right\} dR$$

a minimum value? The first necessary condition furnishes an equation which shows that the greater the curvature of C, the greater the normal derivative of z. From this fundamental formula the theorem is derived.

13. Professor Evans extends the Cauchy existence theorem for partial differential equations to integro-differential equations of so-called "static type," i. e., equations in which the variables of differentiation are different from those of integration. The method of generalization adopted, which is that of the algebra of permutable and non-permutable functions, permits the partial extension of the idea of characteristics to equations of this kind. These curves are given in terms of the variables of differentiation, and are called ordinary characteristics. For special equations every curve in the plane may be an ordinary characteristic; but equivalent equations may sometimes be found whose characteristics are defined. If a curve is not tangent to an ordinary character-

istic, analytic values of the function and its proper derivatives, assigned on the curve, will determine an analytic solution; along a characteristic these values may not be assigned arbitrarily, and do not by themselves determine a unique solution.

14. The ordinary formula of trigonometric interpolation for a function $f(x)$ of period 2π gives a trigonometric sum of order n which coincides in value with $f(x)$ at $2n + 1$ points evenly distributed over a period. The formula is closely analogous to that for the partial sum of the Fourier series for $f(x)$, and the convergence properties of the interpolating function, as n becomes infinite, are similar to those of the Fourier sum. In particular, the mere continuity of $f(x)$ is not a sufficient condition for convergence. Dr. Jackson shows that an interpolating function suggested by Fejér's arithmetic mean in the theory of Fourier's series does always converge, uniformly, if $f(x)$ is continuous; it is a trigonometric sum of order $n - 1$, taking the values of the given function at n equidistant points. The formula which he introduced, for the sake of the rapidity of its convergence in certain cases, in a paper presented at the February meeting of the Society, also converges for every continuous function $f(x)$, but does not coincide exactly with $f(x)$ at the given points, as the present one does. The new formula is examined as to its convergence for certain classes of discontinuous functions; for convergence at any one point, for example, it is sufficient that $f(x)$ be continuous at that point and remain finite everywhere. A simple transformation gives a formula of polynomial interpolation with given points unequally spaced.

15. It has been shown by Poincaré that if two complexes which define the same manifold be made up of a finite number of analytic pieces, their Betti numbers and coefficients of torsion are the same. In this paper, Mr. Alexander proves the invariance of the Betti numbers, the coefficients of torsion, and the numbers which are obtained by reducing the matrices of the complex modulo 2, without assuming the analytic character of the complexes.

16. It is the purpose of Dr. Rowe's first paper to present and illustrate a method of attacking certain problems in regard

to the sums of powers of integers. The four important theorems follow.

(1) The difference of the nth powers (n an odd number > 1) of two odd integers cannot be the kth power (k an integer $\geqq n$) of an integer if the difference of the odd integers is different from zero mod 2^n.

(2) The sum of the nth powers of two odd integers cannot be the kth power of an integer if the sum of the odd integers is different from zero mod 2^n.

(3) The difference of the nth powers of two odd numbers, one of the form $4r + 1$, and the other of the form $4r - 1$, cannot be the k'th power (k' an integer > 1) of an integer.

(4) The sum of the nth powers of two odd numbers, both of the form $4r + 1$, or both of the form $4r - 1$, cannot be the k'th power of an integer.

Applied to Fermat's theorem, theorems (3) and (4) above invalidate one half of all possible solutions of the equation $x^n + y^n = z^n$. Theorems (1) and (2) together with the work of Kummer are shown to prove the impossibility of a solution of $x^n + y^n = z^n$ in integers for all values of n so long as $z(z > x, z > y)$ is a number not greater than 10^{29}.

Further, the theorems of the paper are important quite apart from their application to Fermat's theorem.

This paper has been published in the Johns Hopkins Circular, July, 1913.

17. Dr. Rowe's second paper is concerned with the applications of the theorems of the first paper to more restricted sets of integers. Possibly the most important result is that the first two theorems of the first paper together with Dickson's work (*Quarterly Journal,* volume 40) show that there is no solution of the equation $x^n + y^n = z^n$ in integers x, y and z all prime to n for any value of n if z (the greatest of the integers) is an integer of not more than 2,063 digits.

The one theorem of the paper which is in a different class from the others is: The sum of even powers of two odd integers cannot be the Sth power (S an integer > 1) of an integer.

18. In plane and solid projective geometry the infinite region is a line and a plane respectively; in the real geometry of inversion, a point. Much less familiar are the infinite re-

gions for other geometries, for instance in the geometry of inversion in the complex plane. Here the infinite region consists of two ideal minimal lines, one of each set; and in the geometry of inversion of complex three dimensional space it consists of a minimal cone. In the real or complex plane of analysis (cf. Osgood, *Transactions*, volume 13 (1912), page 159) the infinite region consists of the two real straight lines $x = \infty$ and $y = \infty$; and in the three dimensional space of analysis, of three planes. These facts are elaborated in Professor Bôcher's paper, and the attempt is made to make it clear that these infinite regions are just as useful in their proper places as the more familiar line and plane at infinity of projective geometry in theirs.

19. Professor Osgood's first paper contains the following theorems.

Theorem A. If $f(z_1, \cdots, z_n)$ is analytic in every point of the coordinate axes in the space of analysis, then f is a constant.

Theorem A′. If $f(z_1, \cdots, z_n)$ is analytic in those points of the infinite region of the space of analysis which correspond to any $n - 1$ north poles combined with any point whatever of the nth sphere, then f is a constant.

Corollary. If $f(z_1, \cdots, z_n)$ is analytic in every point of the infinite region of the space of analysis, then f is a constant.

Theorem B. If $f(z_1, \cdots, z_n)$ is analytic at all points outside a fixed hypersphere

$$x_1^2 + y_1^2 + x_2^2 + \cdots + y_n^2 > G,$$

and if f is finite in this region, then f is a constant.

20. Professor Osgood's second paper calls attention to the behavior at infinity of integrals $\int (Pdx + Qdy)$ on an algebraic surface $f(x, y, z) = 0$, which are everywhere finite there. Some integrals of this class are analytic at infinity in the space of analysis, others are analytic at infinity in projective space.

21. The purpose of Professor E. H. Moore's preliminary communication is to indicate an instance of his general Fredholm theory of linear integral equations on the basis Σ_4 (cf. BULLETIN, April, 1912) in which the independent variable p

of the functions involved is itself a function π, the functions thus becoming functional operations on the variable function π.

22. In the *American Journal*, volume 31 (1909), page 396, is given a definition of "α is between $\beta\gamma$" in terms of sameness of sense of tetrahedra. On page 388, l. c., is given an analogous definition based on the relation righthandedness of tetrahedra. On page 366, l. c., it is noted that in addition to the relations "sameness of sense" and "righthandedness" the relations $\alpha_1\alpha_2 S\beta_1\beta_2 \cdots \beta_p$, $\alpha_1\alpha_2 \cdots \alpha_p B\beta_1\beta_2$, and $\alpha_1 I\beta_1\beta_2 \cdots \beta_{p+1}$ also generate n-dimensional geometry. This is evident at once, if it is recognized that corresponding definitions of betweenness are given, by employing as a guide the definition in terms of sameness of sense. Mr. Schweitzer points out that the five definitions of betweenness indicated have the common characteristic that they define affirmatively a linear conception in terms of a spatial conception. The problem of phrasing corresponding geometric systems so that the basal relations function most naturally in each instance is considered in a memoir "A theory of geometrical relations" which is not yet concluded.

23. Using his "Foundations of Grassmann's extensive algebra," *American Journal* (1913), pages 37–56, as a basis Mr. Schweitzer deduces in detail the theory of linear vectors, The first part of the paper is concerned with the deduction of the relation $\beta - \beta' = q(\alpha - \alpha')$, where q is any number, and its fundamental properties; a feature is the use of the author's definition of parallel lines, l. c. In the second part the relation

$$(a + b + c + d)\xi = a\alpha + b\beta + c\gamma + d\delta,$$

where a, b, c, d are any numbers, is derived. In the third part the regressive product of two vectors is defined, also their metrical properties such as congruence, length, etc. The concluding part investigates to what extent the author's fundamental properties (l. c.) induce the equality of domains of points. This part represents essentially an application of Grassmann's statement:

"Die Gleichheit zweier Theile eines Elementarsystems besteht im allgemeinsten Sinne darin, dass beide von dem in einfachem Sinne erzeugten Systeme von Elementen gleiche Gebiete umfassen, nämlich so, dass wechselseitig jedem Element des einen Gebietes ein, aber auch nur ein Element des andern entspricht.

24. In Mr. Schweitzer's third paper the following theorems on Grassmann's synthetic and analytic associations are proved:

I. $\{(y \cup z) \cap z\} = y$ and $\{(y \cap z) \cup y\} = z$ imply $(y \cap z) = (z \cap y)$ and $\{y \cup (y \cup z)\} = z$.

II. $\{y \cup (x \cup z)\} = \{z \cup (x \cup y)\}$ and $\{(y \cup z) \cap z\} = y$ and $\{(y \cap z) \cup y\} = z$ imply $\{y \cap (x \cap z)\} = \{z \cap (x \cap y)\}$.

III. $\{y \cap (x \cap z)\} = \{z \cap (x \cap y)\}$ and $\{(y \cup z) \cap z\} = y$ and $\{(y \cap z) \cup y\} = z$ imply $\{y \cup (x \cup z)\} = \{z \cup (x \cup y)\}$.

As an application, theorems on the author's functional relations are obtained by replacing the associations $(x \cap y)$ and $(x \cup y)$ by $\phi(x, y)$ and $f(x, y)$ respectively. Finally, the problem of finding the quasi-transitive function $f_1(x, y)$ such that

$$f_1\{f(x, y), z\} = f\{f_1(x, z), f_1(y, z)\},$$

where $f(x, y)$ is quasi-transitive, is considered. An actual and not merely formal solution is obtained by using certain well known functional relations due to Abel and Schroeder.

25. The uniformity of convergence of series is justly considered one of the most important conceptions that underlie analysis. Mr. Schweitzer first reviews the manner in which this conception enters into various analytic disciplines; then discusses its relation to the general concept *iteration* and finally takes account of uniformity of convergence by means of his "principle of continuation," which is stated thus:

"The existence of a class of particular elements (or operands) affected by particular operations implies the existence of a class of general elements affected by general operations."

Two special applications of this principle must here be noted, viz., (1) invariance of domain of elements under generalization of operations; and (2) persistence of operational properties under generalization of operands.

26. Professor Kasner shows that for imaginary analytic curves the limit of the ratio of the arc to the chord may not be

unity. The limit is always real, and if it is not unity is at most equal to $\frac{2}{3}\sqrt{2} = .94 \cdots$. A different result is obtained for irregular (singular) points. An extension is made to space.

27. Under the postulate that each real number is equally likely a priori to be the true value, the attempt has been made to deduce from the Gaussian law the principle that the arithmetic mean of n measurements is a posteriori the most probable value of the unknown true value. But this postulate has already* been shown to be untenable. Dr. Dodd investigates several postulates, and finds each of the following three postulates to be inadequate: (1) The a priori probability is constant in a certain interval, and is zero outside this interval. (2) The a priori probability is continuous, and is zero outside a certain interval. (3) The a priori probability is continuous for all real values and does not vanish.

But if the a priori probability is such that the derivative of its logarithm is numerically less than some constant for all real values, then a posteriori under the Gaussian law, after n measurements have been made, the arithmetic mean of these measurements differs from the most probable value by an infinitesimal, as n increases indefinitely. Other conditions less restrictive are found under which the arithmetic mean approximates the most probable value.

28. The linear complexes which osculate the two asymptotic curves of a surface point at their point of intersection have in common a linear congruence of whose directrices one passes through the surface point while the other lies in the corresponding tangent plane. If these directrices be constructed for every point of a surface, there arise two congruences, the directrix congruences of the surface. There exist, in general, two one-parameter families of curves on the surface such that the directrices of the points of such a curve form developables. Professor Wilczynski's paper is devoted to the exceptional case where these so-called directrix curves are indeterminate, i. e., where the directrices of all points of an arbitrary curve on the surface form developables. It is shown that every nonruled surface of this kind is a projective transformation of an integral surface of the completely integrable system

* This BULLETIN, June, 1913, p. 481.

$$\frac{\partial^2\theta}{\partial u^2} + \frac{\varphi_u}{\varphi}\frac{\partial\theta}{\partial u} + 2\varphi\frac{\partial\theta}{\partial v} = 0,$$

$$\frac{\partial^2\theta}{\partial v^2} + 2\varphi\frac{\partial\theta}{\partial u} + \frac{\varphi_v}{\varphi}\frac{\partial\theta}{\partial v} = 0,$$

where θ is a non-vanishing constant or any solution of the partial differential equation

$$\frac{\partial^2 \log \varphi}{\partial u \partial v} = 4\varphi^2 + \frac{k}{\varphi},$$

where k is a constant, which is or is not equal to zero according as the corresponding surface has asymptotic curves which do or do not belong to linear complexes. In the former case, the explicit equations of the surface can be found by quadratures.

One of the directrix congruences of any surface of this class consists of all of the lines through a fixed point, while the second directrix congruence is composed of all of the lines of a fixed plane. These are called the directrix point and directrix plane of the surface, and are united in position if and only if $k = 0$.

If the asymptotic curves of the surface are projected from the directrix point on the directrix plane, there is obtained a net of plane curves which has very remarkable properties. It is periodic of period 3 under Laplace transformation, and the two nets distinct from it obtained by Laplace transformation are so related as to have at each pair of corresponding points a common osculating conic. These properties are characteristic; i. e., any net of plane curves which has these properties may be regarded as a projection of the asymptotic curves of a surface with indeterminate directrix curves. A net of this kind is determined except for three arbitrary constants by means of one curve of each of its two families, and these may be assigned arbitrarily. There exists a single infinity of surfaces with indeterminate directrix curves which correspond to a given net of this kind and a given point as directrix point.

Incidentally a complete geometrical interpretation is obtained for the equality of the invariants h and k of a single partial differential equation of the form

$$\frac{\partial^2\theta}{\partial u \partial v} + a\frac{\partial\theta}{\partial u} + b\frac{\partial\theta}{\partial v} + c\theta = 0,$$

a case which has received considerable attention especially from Darboux and Moutard, but whose geometrical significance apparently has not been discussed so far.

29. In this paper Professor Shaw calls attention to the necessity of considering the axes of a linear vector operator (which are foreign to a matrix). For the transverse the shear regions are made up of vectors which are conjugate to those defining the shear regions of the original linear vector operator, but respectively reversed in order. In the case of self-transverse operators when the multiplicity of any root is $m > 1$ the shear regions are representable in the forms

$$\begin{aligned} &\psi(i_1 + \sqrt{-1}j_m), \quad \psi(i_2 + \sqrt{-1}j_{m-1}), \cdots \\ &\psi(i_r + \sqrt{-1}j_r), \quad \psi(i_r - \sqrt{-1}j_r), \cdots \\ &\psi(i_1 - \sqrt{-1}j_m). \end{aligned}$$

In this $r = \frac{1}{2}m$ for m even, $\frac{1}{2}(m + 1)$ for m odd. In case m is odd $j_r = 0$. The vectors i_s, j_s form a mutually orthogonal system, and ψ has an inverse.

30. Professor Cajori's paper is a study of the purpose of Zeno's arguments as it is reported by Plato and as it is set forth by modern writers, particularly by E. Zeller, W. Windelband, M. Cantor, V. Cousin, G. Grote, P. Tannery, G. Milhaud, P. Natorp, and J. Cohn.

31. Professor Glenn defines and applies a translation principle by which every invariant of a plane n-line corresponds to a contravariant of a definite line congruence in 3-space. The idea of a tangential equation of the congruence is developed from the invariantive point of view.

32. Ciani (*Rendiconti di Palermo*, 1899; *Annali di Matematica*, 1901) has discussed the systems of conics through 8 inflections on the various self-projective quartics. In this paper Professor Sharpe confirms Ciani's results by a different method and also proves the existence of systems of conics through 6 inflections,

the Klein quartic having 2100 such conics in addition to the 112 found by Ciani.

33. The existence of irreducible plane curves with any number $\leqq \frac{1}{2}(n-1)(n-2)$ of distinct double points was proved by Snyder (this BULLETIN, volume 15, page 1, 1908). Guccia (*Supplemento Rendiconti Palermo*, volume 5, page 1, 1910) proposed the problem of the maximum number of consecutive double points for an irreducible plane curve of order n. In this paper Professor Sharpe and Dr. Craig show that the maximum number is at least the greatest integer in $\frac{1}{4}(n^2+3n-14)$. The rational sextics with consecutive double points are shown to be projectively equivalent and their equation in a canonical form is determined.

34. In this paper Miss Sanderson shows a way of constructing modular covariants of a binary form similar to that in which ordinary covariants are constructed as symmetric functions of the roots. A form of the nth degree is written as the product of n linear factors, the ratios of whose coefficients are the roots of the form when it is equated to zero. It proves simpler to construct the covariants as functions of these coefficients rather than as functions of the roots. It is found that many different types of symmetric functions may enter in addition to the two simple types which suffice for the ordinary covariants. Some covariants are found to be easily obtainable, as in the ordinary theory, from the coefficient of the leading term.

35. By means of uniform sets of "superpostulates" Dr. Sheffer is enabled (1) to determine, by one type of formula, the class of deductive systems having a postulated property; (2) to determine completely, by the same type of formula, a given deductive system; (3) after the reduction of an ordinary postulate set to its corresponding superpostulate set, to solve by inspection (*a*) the problems of consistence, independence, and one-valuedness, (*b*) a new problem, that of internal independence.

36. Dr. Sheffer determines Boolean algebras by six independent postulates involving a non-commutative operation. The existence of the special element u is postulated; that of z and of a' is proved.

37. In a former paper, to be published in the *American Journal*, Professor Winger tabulated the varieties of self-projective rational curves of the fourth and fifth orders. In the present paper he considers the case of the rational sextic, exhibits the types associated with the different groups, and points out some of the more immediate inferences.

There are one or more projectively distinct curves attached to each of the following groups: cyclic G_n, dihedral G_{2n} $(2 \leqq n \leqq 6)$, the tetrahedral G_{12}, the octahedral G_{24}, the icosahedral G_{60}, and an infinite group. Thus the sextic is the first curve to admit the tetrahedral group (simply) or the icosahedral, the group of maximum order for rational curves.

38. This is a continuation of the preceding paper. The following general theorem is proved: A rational plane curve C^m of odd order cannot admit the tetrahedral G_{12}, the octahedral G_{24}, the icosahedral G_{60}, nor a dihedral G_{2n}, n even. Combined with a former theorem, this limits the types of septimics to those invariant under cyclic groups of orders 2, . . . , 7 and dihedral groups of orders 6, 10, and 14. All types are found to occur.

39. Following J. Hadamard, Professor Fréchet criticizes the definition of derivatives in functional calculus given by V. Volterra. He shows that one definition of differentials must be given before that of derivatives, and gives a new definition of differentials in general analysis. He then proves that the differential of a function U_L of a line L can be written

$$\int_a^b (y_0'\Delta z - z_0'\Delta y)dp + \int_a^b (z_0'\Delta x - x_0'\Delta z)dq + \int_a^b (x_0'\Delta y - y_0'\Delta x)dr$$
$$+ R[x_0'(a)\cdot\Delta x(a) + y_0'(a)\cdot\Delta y(a) + z_0'(a)\cdot\Delta z(a)]$$
$$+ S[x_0'(b)\cdot\Delta x(b) + y_0'(b)\cdot\Delta y(b) + z_0'(b)\cdot\Delta z(b)],$$

where x_0', y_0', z_0' are the derivatives of the coordinates $x_0(t)$, $y_0(t)$, $z_0(t)$ on L_0, $(a \leqq t \leqq b)$; $\Delta x(t)$, $\Delta y(t)$, $\Delta z(t)$ are the projections of the displacement; $A, B, C;$ A', B', C' are constants; $p(t)$, $q(t)$, $r(t)$ are functions with limited total fluctuation, and where the integrals are Stieltjes's integrals. The paper will appear in the *Transactions*.

40. Under the assigned conditions of Professor Laves's paper the system of two satellites is assumed to be placed at the time t_0 at such a distance from the central mass that the sum of the angles of recession of the nodes of the two orbits on the equatorial plane of the central mass equals twice the angle of elongation. The angles of inclination γ and γ' are considered small quantities of the first order at the time t_0. The following theorem is obtained: Even if the original inclinations of the orbits are small, by the mutual perturbations of the satellites additional terms are created of a librational character. The "imparted" inclination of each satellite contains four factors: (1) the mass of the perturbing satellite, (2) its inclination, (3) the ratio of the mean daily motion n of the perturbed body to the mean daily motion of the point of conjunction with the perturbing satellite, (4) $\frac{1}{1-\kappa}$, where $\kappa = \frac{\beta n + \beta' n'}{4n' - 2n}$; βn and $\beta' n'$ measure the recession of the line of nodes of each body upon the equator.

The close proximity of κ to unity makes the librational term in the case of one of Saturn's inner system of satellites of primary importance. The theorem explains the existence of considerable inclinations in this system, first discovered by H. Struve.

41. Let G be a primitive group of linear homogeneous substitutions of determinant unity in n variables. In Professor Blichfeldt's paper the following points are noted: (1) The order of an abelian subgroup of G of variety m is $\leqq 5^{m-1}$ if it is free from similarity substitutions. If the order is at the same time a power of a prime $p \neq 2$ or 5, then it is $\leqq 4^{n-1}$. (2) If the order of a Sylow subgroup of G be p^λ, where $p > 3$, $> n/2$, and $\neq n$, then that subgroup is abelian. (3) Let these subgroups be arranged as follows:

$\frac{1}{2}n < p_1 \leqq \frac{1}{2}n + 1$, the order being $p_1^{\lambda_1}$;

$\frac{1}{2}n + 1 < p_i \leqq n + 1$, the order being $p_i^{\lambda_i}$

$$(i = 2, 3, \cdots, a;\ p_i \neq n);$$

$n + 1 < p_j$, the order being $p_j^{\lambda_j}$ $\qquad (j = a + 1, \cdots, b)$.

Then there is in G an abelian subgroup of order

$$p_1^{[\lambda_1]} \cdot \prod_{i=2}^{a} p_i^{\lambda_i - 1} \cdot \prod_{j=a+1}^{b} p_j^{\lambda_j},$$

where $[\lambda_1] = 0$ or $\lambda_1 - 1$, according as $\lambda_1 \leqq 2$ or > 2. (4) The order of G must be $< \frac{1}{2}n[(n+1)!] \cdot 5^{2(n-1)} \cdot 4^{g(n-1)}$, where g represents the number of primes that are $\geqq 2$ and $\leqq n/2$.

42. In Professor Running's paper different values are assigned to y in the equation

$$f(dy/dx, x, y) = 0$$

and curves corresponding to the resulting equations are traced with dy/dx as ordinate. Starting from the point represented by the initial condition, points on the curves are joined in such a way as to make the area under each connecting line equal to the difference between corresponding values of y. The derived curve is approximated to by the connecting lines. The coordinates of points on the integral curve are read from the figure. The method is applied to differential equations of the nth order where the initial conditions are represented by the first $n - 1$ derivatives at a given point.

The first part of this paper will appear in the November number of the *American Mathematical Monthly*.

43. In a previous paper Professor Baker showed that the genus of a Cayley diagram representing a group might fall below the Dyck genus if the group were abelian. In the present paper an example of reduction is shown to occur in the case of three generators not abelian, namely, the genus of G_{32}, Burnside, page 81, XIV, $m = 2$, can be reduced to 10, the Dyck genus being 13. For diagrams not having self-touching polygons the genus is not less than the sum of the genera of the generational "Nebengruppe," that is, those lines of a rectangular table of the group which are extensions of a subgroup by the power of the remaining generator.

44. Professor Baker's second paper is in abstract as follows: The $S(3, 2, 7)$ can be represented by 7 points, 21 arcs, and 14 triangles on an anchor ring. Seven of the triangles may have the points of the lines of the system, and the other seven

the points of the lines of another system; or the arrangement of the seven lines may appear as one triangle and six routes. This alternative is general for all the systems $S(s+1, 2, s^2+s+1)$. The $S(5, 2, 21)$ can be exhibited by a configuration of 21 pentagons, corresponding to the columns of the system, and 21 skew pentagons. The genus is 22 and the arrangement is invariant under a rotation of $2\pi/21$. It is possible so to choose the notation that the three real points and the two Galois imaginary points on each line are simultaneously separated.

The $S(s+1, 2, s^2+s+1)$, s an odd prime, is not representable by a simple M_2. The polygons representing the lines fall into at least two cycles at a point. For $s(4, 2, 13)$ the representation may be made by a body with one ordinary hole and thirteen holes only joined up at a pinch point. At these pinch points eight quadrilaterals come together in two sets of four, two of the system and two of an auxiliary system in each.

To represent further invariances under the group it is necessary to use more complicated manifolds. For example, tetrahedra hanging together on edges, in pairs in an R_3 not of simple connectivity in an R_4, are needed for the case $S(4, 2, 13)$.

45. In this paper Professor Carmichael proves several general theorems concerning the positive or negative integers x, y, z and the integers n and k (each greater than unity) which satisfy the equation

$$x^n - y^n = z^k.$$

Typical of these results is the following: the greatest divisor d of $x - y$ which is prime to x, y and n is a perfect kth power. In addition he proves the following special theorems: the difference of the fourth powers of two relatively prime integers cannot be the cube of an integer; neither the sum nor the difference of the sixth powers of two integers can be the square of an integer.

46. In the first part of Professor March's paper it is shown that the double integral representation of an arbitrary function in terms of spherical harmonics,

$$f(\theta) = \int_0^\infty \alpha d\alpha P_{\alpha-\frac{1}{2}}(\cos\theta) \int_0^\pi f(\gamma) S_{\alpha-\frac{1}{2}}(\cos\gamma)\sin\gamma d\gamma,$$

established in his Munich dissertation,* "Ueber die Ausbreitung der Wellen der drahtlosen Telegraphie auf der Erdkugel," for the case where $f(\theta)$ is continuous in the interval $(0, \pi)$ and is subject to certain other restrictions, is also capable of representing a function which has a finite number of finite discontinuities in this interval, and that at such a point of discontinuity the integral represents the mean of the two limiting values of the function.

In the second part of the paper use is made of the method employed in establishing the integral representation to obtain the well-known development of an arbitrary function in a series of spherical harmonics. The method has the advantage of not requiring a proof of the possibility of the development, as this is made to depend upon the possibility of the development of an arbitrary function in a Fourier series of special form.

H. E. SLAUGHT,
Acting Secretary.

INTUITIONISM AND FORMALISM.†

BY DR. L. E. J. BROUWER.

(Inaugural address at the University of Amsterdam, read October 14, 1912.)

THE subject for which I am asking your attention deals with the foundations of mathematics. To understand the development of the opposing theories existing in this field one must first gain a clear understanding of the concept "science"; for it is as a part of science that mathematics originally took its place in human thought.

By science we mean the systematic cataloguing by means of laws of nature of causal sequences of phenomena, i. e., sequences of phenomena which for individual or social pur-

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* See note in *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 20 (1911), pp. 353–363.

† Translated for the BULLETIN by Professor ARNOLD DRESDEN.