

ON CLOSED CONTINUOUS CURVES.

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1. In a paper soon to be published it will be proved that in every closed convex curve which is analytic throughout at least one square may be inscribed. Conversely, if in a Cartesian plane an arbitrary square is given, the problem is to find the parametric equations of any convex curve through the vertices of the given square. It is the purpose of this paper to establish these equations.

2. First assume the square $A_1A_2A_3A_4$ symmetric with respect to the x - and y -axes and on the circle

$$(1) \quad x = \frac{a}{2} \sqrt{2} \cos \frac{2\pi t}{w}, \quad y = \frac{a}{2} \sqrt{2} \sin \frac{2\pi t}{w},$$

where t is the parameter and w the period. The coordinates of A_1, A_2, A_3, A_4 are in the same order $(a/2, a/2)$; $(-a/2, a/2)$; $(-a/2, -a/2)$; $(a/2, -a/2)$, and the corresponding parameters $t_k = (2k + 1) \cdot w/8$, ($k = 0, 1, 2, 3$). Designating by $\phi(t)$, $\psi(t)$ two uniform continuous functions for all values of t , and with the same period w , then

$$(2) \quad \begin{aligned} x &= \frac{a}{2} \sqrt{2} \cos \frac{2\pi t}{w} + \lambda \sin \left(\frac{2\pi t}{w} - \frac{\pi}{4} \right) \sin \left(\frac{2\pi t}{w} - \frac{3\pi}{4} \right) \phi(t), \\ y &= \frac{a}{2} \sqrt{2} \sin \frac{2\pi t}{w} + \mu \sin \left(\frac{2\pi t}{w} - \frac{\pi}{4} \right) \sin \left(\frac{2\pi t}{w} - \frac{3\pi}{4} \right) \psi(t),* \end{aligned}$$

where λ and μ are arbitrary constants, represent a closed curve through the vertices of the square. Now any closed continuous curve† may be represented parametrically by

$$(3) \quad x = F(t), \quad y = G(t),$$

* In this expression it is sufficient to take the product of two sines as indicated, not as in the abstract of the paper which appeared in the BULLETIN of February, pp. 221-222, where a product of four sines was introduced.

† Osgood: Lehrbuch der Funktionentheorie, vol. 1, 2d ed., pp. 146-150.

in which F and G are of the same general type as ϕ and ψ . Hence any closed curve through $A_1A_2A_3A_4$ may be represented in the form (3). We simply have to choose $F(t)$ and $G(t)$ in such a manner that for $t = (2k + 1) \cdot w/8$, $k = 0, 1, 2, 3$, they assume the same values as x and y in (1) and (2). Supposing that $F(t)$ and $G(t)$ satisfy this condition, it must be shown that $\phi(t)$ and $\psi(t)$ in (2) can be determined as continuous co-periodic functions of t , defined for all values of t , such that the right-hand members of (2) become respectively $F(t)$ and $G(t)$. Evidently for all values of t , except $t \equiv (2k + 1) \cdot w/8 \pmod{w}$,

$$(4) \quad \begin{aligned} \phi(t) &= \frac{F(t) - \frac{a}{2}\sqrt{2} \cos \frac{2\pi t}{w}}{\lambda \sin\left(\frac{2\pi t}{w} - \frac{\pi}{4}\right) \sin\left(\frac{2\pi t}{w} - \frac{3\pi}{4}\right)}, \\ \psi(t) &= \frac{G(t) - \frac{a}{2}\sqrt{2} \sin \frac{2\pi t}{w}}{\mu \sin\left(\frac{2\pi t}{w} - \frac{\pi}{4}\right) \sin\left(\frac{2\pi t}{w} - \frac{3\pi}{4}\right)}. \end{aligned}$$

For all values of t , different from odd multiples of $w/8$, $\phi(t)$ and $\psi(t)$ as defined by (4) are continuous and well defined. For $t = (2k + 1) \cdot w/8$ both $\phi(t)$ and $\psi(t)$ become indeterminate. We find, however,

$$\begin{aligned} \lim_{t \equiv (2k+1)w/8} \{\phi(t)\} &= \lim_{t \equiv (2k+1)w/8} \left\{ \frac{F'(t) + \frac{a\sqrt{2} \cdot \pi}{w} \sin \frac{2\pi t}{w}}{\frac{\lambda 2\pi}{w} \sin\left(\frac{4\pi t}{w} - \pi\right)} \right\} \\ &= \frac{F'(t) + \frac{a\pi\sqrt{2}}{w} \sin(2k+1)\frac{\pi}{4}}{\frac{\lambda 2\pi}{w} \sin(2k-1)\frac{\pi}{2}} \\ &= \pm \frac{wF'(t) \pm a\pi}{2\lambda\pi}, \end{aligned}$$

which is evidently a finite quantity. The same is true of $\lim \{\psi(t)\}$ for the same values of t .

From this follows that $\phi(t)$ and $\psi(t)$ as defined by (4) are continuous for all values of t and, consequently, that any

closed analytic curve through the vertices of the given square may be parametrically represented by (2).

3. Applying to the points (x, y) of the Cartesian plane a combined rotation and translation θ, p, q , so that after the motion the coordinates of the displaced points with respect to the same plane are

$$(5) \quad \begin{aligned} x' &= p + x \cos \theta - y \sin \theta, \\ y' &= q + x \sin \theta + y \cos \theta, \end{aligned}$$

and choosing the side a of the square properly, the transformed square $A_1'A_2'A_3'A_4'$ may represent any square in the plane. Putting

$$(6) \quad \begin{aligned} \lambda\phi(t) \cos \theta - \mu\psi(t) \sin \theta &= f(t), \\ \lambda\phi(t) \sin \theta + \mu\psi(t) \cos \theta &= g(t), \end{aligned}$$

the parametric equations of the closed curve through the square in the new position, after substituting in (5) for x and y their expressions as given in (2), and reducing, may be written

$$(7) \quad \begin{aligned} x' &= p + \frac{a}{2} \sqrt{2} \cos \left(\frac{2\pi t}{w} + \theta \right) \\ &\quad + \sin \left(\frac{2\pi t}{w} - \frac{\pi}{4} \right) \sin \left(\frac{2\pi t}{w} - \frac{3\pi}{4} \right) f(t), \\ y' &= q + \frac{a}{2} \sqrt{2} \sin \left(\frac{2\pi t}{w} + \theta \right) \\ &\quad + \sin \left(\frac{2\pi t}{w} - \frac{\pi}{4} \right) \sin \left(\frac{2\pi t}{w} - \frac{3\pi}{4} \right) g(t). \end{aligned}$$

In (6) $\phi(t)$ and $\psi(t)$ may always be determined in such a manner that $f(t)$ and $g(t)$ are any two distinct continuous and co-periodic functions of t .

A closed analytic curve through the vertices of a square may therefore always be represented by the parametric equations (7).

As it is always possible to inscribe at least one square in an analytic convex curve* (ordinary oval), the parametric equations of such a curve may always be written in the form (7).

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* The proof for this theorem will appear in the *American Journal of Mathematics*. Recently I have been able to prove that the theorem holds for any closed analytic curve without singular points.