SOME PROPERTIES OF SPACE CURVES MINIMIZING A DEFINITE INTEGRAL WITH DISCONTINUOUS INTEGRAND.

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IN a recent paper Bliss and Mason* considered the problem of the calculus of variations in which the integrand function is allowed to have a finite discontinuity along a given plane curve which separates the fixed end points. Later they made a systematic extension of the Weierstrassian theory of the calculus of variations to problems in space.† The object of the present note is to state the results obtained by applying the method used in the first mentioned paper to the case of a discontinuous integrand occurring in the space problem.

The problem studied may then be stated in the following way: Among all curves which go from the point 1 to the point 3 lying on opposite sides of a given surface S, and which cross S but once, it is required to find the one which minimizes the sum of the two integrals

$$J = \int F(x, y, z, x', y', z')dt, \quad j = \int f(x, y, z, x', y', z')dt,$$

the first integral to be taken from the point 1 to the surface S and the second from S to the point 3.

§ 1. *Equations Defining the Minimizing Curves.*

In order to find the equations of the minimizing curves

$$(1) \qquad x = x(t), \quad y = y(t), \quad z = z(t),$$

it will be supposed that all curves considered lie in the interior of a region R of space. The surface S defined by the equations

$$S: \qquad x = x(u, v), \quad y = y(u, v), \quad z = z(u, v)$$

* Bliss and Mason: "A problem of the calculus of variations in which the integrand is discontinuous," *Transactions*, vol. 7 (1906).

† Bliss and Mason: "The properties of curves in space which minimize a definite integral," *Transactions*, vol. 9 (1908). For brevity this paper will hereafter be referred to as I.

is supposed to have no singular points in R and divides this region into two regions R_1 and R_2 which contain the points 1 and 3 respectively. In R_1 the variables and their derivatives will be denoted by capital letters while in R_2 small letters will be used. Then in the whole of R it will be assumed that the two functions F and f have the properties ordinarily imposed in the space problem. Hence the functions (1) must satisfy the Euler differential equations and if the parameter is chosen as the length of arc and the problem assumed to be regular it follows that the extremals for J can therefore be written in the form

$$(2) \qquad \begin{aligned} X &= \Phi(s; X_1, Y_1, Z_1, X_1', Y_1', Z_1'), \\ Y &= \Psi(s; X_1, Y_1, Z_1, X_1', Y_1', Z_1'), \\ Z &= \mathrm{X}(s; X_1\ Y_1, Z_1, X_1', Y_1', Z_1'), \end{aligned}$$

where

$$(3) \qquad X_1'^2 + Y_1'^2 + Z_1'^2 = 1$$

and the following initial conditions are satisfied

$$(4) \qquad \begin{aligned} &\Phi(0; X_1, Y_1, Z_1, X_1', Y_1', Z_1') = X_1, \\ &\qquad \Phi_s(0; X_1, Y_1, Z_1, X_1', Y_1', Z_1') = X_1', \\ &\Psi(0; X_1, Y_1, Z_1, X_1', Y_1', Z_1') = Y_1, \\ &\qquad \Psi_s(0; X_1, Y_1, Z_1, X_1', Y_1', Z_1') = Y_1', \\ &\mathrm{X}(0; X_1, Y_1, Z_1, X_1', Y_1', Z_1') = Z_1, \\ &\qquad \mathrm{X}_s(0; X_1, Y_1, Z_1, X_1', Y_1', Z_1') = Z_1'. \end{aligned}$$

A similar set of extremals exists for the integral j which can be denoted by the equations

$$(5) \qquad \begin{aligned} x &= \varphi(s; x_1, y_1, z_1, x_1', y_1', z_1'), \\ y &= \psi(s; x_1, y_1, z_1, x_1', y_1', z_1'), \\ z &= \chi(s; x_1, y_1, z_1, x_1', y_1', z_1'), \end{aligned}$$

where the functions φ, ψ, χ satisfy initial conditions similar to (4).

Let 2 be the point where the minimizing curve joining 1 and 3 intersects the surface S. Then since the arcs C_{12} and C_{23} must minimize the integrals J and j with respect to all curves joining 1 and 2, 2 and 3 respectively and lying in the regions R_1 and R_2 it follows that the regular conditions I and II are necessary:

I. *The arcs C_{12} and C_{23} are extremals for the integrals J and j, respectively, and therefore belong to the sets* (2) *and* (5).

II. *The functions F_1 and f_1 must be positive along the arcs C_{12} and C_{23} satisfying* I.

§ 2. *The Third Necessary Condition.*

The third necessary condition is found to be a restriction on the direction of the extremal arcs C_{12} and C_{23} at the point of intersection 2 with the surface S.

In order to find what this restriction is let it be supposed that the minimizing curve is imbedded in a one-parameter family of curves

$$(6) \qquad x = a(t, u), \quad y = b(t, u) \quad z = c(t, u),$$

which has the following properties. The curves all intersect the surface S for $t = t_2$, pass through the point 1 for $t = t_1$, through 3 for $t = t_3$, and contain the curve C for $u = u_2$. Thus it has been assumed that the point 2 varies along the u-parameter curve of S. The sum of the two integrals J and j taken along any number of the set (6) from 1 to 3 is evidently a function of u which may be denoted by $I(u)$. But if C is to furnish a minimum the derivative $I'(u)$ must vanish when $u = u_2$. By the ordinary methods of variations this leads to the following result:

$$(7) \qquad [F_{X'} - f_{x'}]x_u + [F_{Y'} - f_{y}]y_u + [F_{Z'} - f_{z'}]z_u = 0,$$

where the arguments of the derivatives of F are the values of X, Y, Z, X', Y', Z' on the curve C_{12} at the point 2, while those of the derivatives of f are the values x, y, z, x', y', z' on C_{23} at 2 and x_u, y_u, z_u define the direction of the u-parameter line on S.

Likewise when the point 2 varies on the v-parameter line of the surface there results the corresponding equation

$$(8) \qquad [F_{X'} - f_{x'}]x_v + [F_{Y'} - f_{y'}]y_v + [F_{Y'} - f_{z'}]z_v = 0.$$

The third condition is then

III. *The extremal arcs C_{12} and C_{23} together with the surface S must satisfy equations* (7) *and* (8) *at the intersection point* 2 *on the surface.*

Since the direction cosines of any curve on S are proportional to

$$x_u u' + x_v v', \quad y_u u' + y_v v', \quad z_u u' + z_v v',$$

it follows that equations (7) and (8) will be satisfied when the point moves on any curve passing through 2 and furthermore the bracketed quantities in these equations are proportional to the direction cosines of the normal to the surface at 2.

§ 3. *The System of Extremals for j Determined by Those of J and Condition III.*

It can now be shown that if a curve C_{123} has been found which satisfies conditions I, II and III then each extremal C_{12}, of the set (2) determines uniquely in connection with III one of the set (5).

In order to see this, think of equations (2) as a set of extremals through the fixed point (X_1, Y_1, Z_1) intersecting the surface at points 2′. X_1', Y_1', Z_1' are then parameters satisfying the equation (3).

In place of the parameters X_1', Y_1', Z_1' it will be found advantageous to introduce the values of the u and v parameter lines of S. This can be accomplished by solving* the equations

$$(9) \qquad \begin{aligned} \Phi(s; X_1, Y_1, Z_1, X_1', Y_1', Z_1') &= x(u, v), \\ \Psi(s; X_1, Y_1, Z_1, X_1', Y_1', Z_1') &= y(u, v), \\ \mathrm{X}(s; X_1, Y_1, Z_1, X_1', Y_1', Z_1') &= z(u, v), \\ X_1'^2 + Y_1'^2 + Z_1'^2 &= 1 \end{aligned}$$

for s, X_1', Y_1', Z_1' in terms of u and v and substituting the values of X_1', Y_1', Z_1' thus obtained in (2). Equations of the following form result

$$(10) \qquad X = \Phi(s, u, v), \quad Y = \Psi(s, u, v), \quad Z = \mathrm{X}(s, u, v).$$

Consider now the equations

$$(11) \qquad \begin{aligned} x_2'^2 + y_2'^2 + z_2'^2 &= 1, \\ f_{x'}x_u + f_{y'}y_u + f_{z'}z_u &= F_{X'}x_u + F_{Y'}y_u + F_{Z'}z_u, \\ f_{x'}x_v + f_{y'}y_v + f_{z'}z_v &= F_{X'}x_v + F_{Y'}y_v + F_{Z'}z_v, \end{aligned}$$

* This is always possible for points 2′ on S sufficiently near 2, provided the point 2 on C_{12} is not conjugate to 1. See Bliss and Mason, I, p. 446 and following.

of which the last two are simply the equations (7) and (8) written in a separated form. By means of the equations of the surface S and (10) it is possible to express X_2, Y_2, Z_2; x_2, y_2, z_2; x_u, y_u, z_u; x_v, y_v, z_v and X_2', Y_2', Z_2' in terms of u and v. Equations (11) are then functions of x_2', y_2', z_2', u and v, having one solution corresponding to 2. Hence there will be a unique solution for x_2', y_2', z_2' as functions of u and v provided the functional determinant of the left hand members of (11) is different from zero.

This determinant is found to have the value*

$$c[x_2'\{f_x' - F_X'\} + y_2'\{f_y' - F_Y'\} + z_2'\{f_z' - F_Z'\}] \qquad f_1(x_2'^2 + y_2'^2 + z_2'^2),$$

an expression which can vanish only when the term in brackets vanishes. This can happen only when the arc C_{23} is tangent to S at 2—a case which will be excluded. Hence there is a unique solution for x_2', y_2' and z_2' in terms of u and v, and when these values are substituted in equations (5) it is seen that they assume the form

$$(12) \qquad x = \varphi(s, u, v), \quad y = \psi(s, u, v), \quad z = \chi(s, u, v),$$

where of course x_2, y_2, z_2 have also been replaced by the functions $x(u, v)$, $y(u, v)$, $z(u, v)$ defining the point 2 on S.

It has therefore been shown that *if C_{123} is a curve satisfying conditions* I, II, III, *then to each extremal through* 1, $C_{12'}$ *of the integral J and near C_{12} there corresponds one extremal of the integral j, which with $C_{12'}$ satisfies the corner condition* III *at* 2′. *The two parameter set of extremals for J thus defines another two parameter set for j with initial points on S. The equations of the two sets with u and v as parameters can be put in the forms* (10) *and* (12)

Furthermore the functional determinant Δ of the functions defined in (12)

$$\Delta(s, u, v) = \begin{vmatrix} \varphi_s & \varphi_u & \varphi_v \\ \psi_s & \psi_u & \psi_v \\ \chi_s & \chi_u & \chi_v \end{vmatrix}$$

is found to be different from 0 for the point 2 and hence from continuity conditions for points near 2.

* For a method of evaluating see Bliss and Mason, I, p. 447.

If however it is assumed that Δ does vanish for some point 4 on C_{23} and that at least one of the three-rowed determinants of the matrix

$$\begin{Vmatrix} \Delta_s & \Delta_u & \Delta_v \\ \varphi_s & \varphi_u & \varphi_v \\ \psi_s & \psi_u & \psi_v \\ \chi_s & \chi_u & \chi_v \end{Vmatrix}$$

does not vanish with Δ, it then follows* that the extremals (12) have an enveloping surface D which touches the curve C_{23} at 4. Moreover it is known from the general theory that there is a single definite curve

$$d: \qquad x = x(\alpha), \quad y = y(\alpha), \quad z = z(\alpha)$$

on S which touches C_{23} at 4 and is the envelope of a one parameter family of extremals selected from (12) and containing C_{23} for $\alpha = 0$. The equation of this last family may be written in the form

$$(13) \qquad x = \varphi(s, \alpha), \quad y = \psi(s, \alpha), \quad z = \chi(s, \alpha).$$

Furthermore it is possible to select from the two-parameter family (10) a one-parameter family involving the parameter α and satisfying the direction conditions III. For in connection with these equations it was shown that any $C_{12'}$ determined uniquely a $C_{2'3}$ with initial point on S. Since however the functional determinants for both sets of extremals are different from zero at 2 it follows that the converse also holds. Consequently when a one-parameter family (13) satisfying the imposed conditions is chosen from (12) there goes with it a definite one-parameter family of the set (10), say

$$(14) \qquad X = \Phi(s, \alpha), \quad Y = \Psi(s, \alpha), \quad Z = \mathrm{X}(s, \alpha),$$

containing C_{12} for $\alpha = 0$.

§ 4. *The Jacobi Condition.*

By means of the two one-parameter families just given and their enveloping curve d it can now be proved that an arc C_{123} which joins 1 to 3 and minimizes the sum of the two integrals J and j cannot have upon it a point of contact 4 with

* See Bliss and Mason, I, p. 449.

the enveloping surface D. For consider the sum of the integral J taken along $C_{12'}$ from 1 to S, plus the value of j taken along $C_{2'3'}$ from S to the contact point $4'$ of $C_{2'3'}$ with d and then, along d from $4'$ to 4. It is found that this sum has a constant value, for its derivative with respect to α is zero.

In fact it is readily verified by the ordinary methods that

$$\frac{dJ}{d\alpha}(C_{12'}) = x_\alpha F_{X'} + y_\alpha F_{Y'} + z_\alpha F_{Z'}/^{2'},$$

where x_α, y_α, z_α are the direction cosines of a line on S. Likewise

$$\frac{dj}{d\alpha}(C_{2'4'}) = -[f_{x'}x_\alpha + f_{y'}y_\alpha + f_{z'}z_\alpha]/^{2'} + f(x, y, z, x_\alpha, y_\alpha, z_\alpha)/^{4'},$$

and

$$\frac{dj(d_{4'4})}{d\alpha} = -f(x, y, z, x_\alpha, y_\alpha, z_\alpha)/^{4'}.$$

Recalling now the direction condition at the point $2'$, it is seen that the sum of these three derivatives is zero and therefore the sum of the three integrals $J(C_{12'})$, $j(C_{2'4'})$ and $j(d_{4'4})$ is constant in value and in particular is equal to the sum of $J(C_{12})$ and $j(C_{24})$.

The usual argument with regard to the envelope d not being a solution of the Euler equations can now be applied, from which it follows that if 4 is not a singular point of d then the arcs C_{12} and C_{23} cannot minimize the sum $J_{12} + j_{23}$ if 4 lies on the arc C_{23}. Therefore as a fourth necessary condition it can be stated that

IV. *The extremal arcs* C_{12} *and* C_{23} *can contain no points conjugate to their initial points* 1 *and* 2. *Therefore the curve* d *on the enveloping surface* D *of the extremals* $C_{2'4'}$ *must not touch the arc* C_{23} *before the point* 3.

§ 5. *Sufficient Conditions.*

Suppose now that a curve C_{123} has been found which satisfies conditions I, II, III and IV strengthened by the assumption that: IV′, the curve d does not touch the arc C_{23} even at the point 3. It is desired to see if C_{123} actually minimizes the sum of the two integrals under these conditions.

It has already been shown that if the arc C_{12} does not contain the point conjugate to 1 it can be imbedded in a two-parameter family of extremals $C_{12'}$ passing through 1. Moreover the functional determinant of this family is different from zero and it follows therefore that the set $C_{12'}$ forms a field F' about C_{12}. Further, on account of condition IV′, the determinant $\Delta(s, u, v)$ of the set $C_{2'4'}$ determined by $C_{12'}$ and condition III is different from zero along C_{23} and so the set $C_{2'4'}$ forms a field F'' about the arc C_{23}. In each of these fields the properties of the extended invariant integral* hold.

Consider then any curve

$$C: \qquad x = x(t), \quad y = y(t), \quad z = z(t)$$

which joins the two fixed points 5 and 7, crosses S once at 6 and lies entirely in the fields F' and F''. It can be proved that the following theorem holds: *The sum of the two integrals* $J^*(C_{56})$ *and* $j^*(C_{67})$ *is independent of the path of integration and depends only on the end points* 5 *and* 7.

In order to prove this consider any comparison curve $\overline{C}$ which joins 5 and 7, lies entirely in the fields F' and F'', and crosses S once at the point 6′. Since each of the integrals is invariant in its respective field the two following equations result

$$J^*(\overline{C}_{56'}) + J^*(k_{6'6}) = J^*(C_{56}),$$

$$j^*(k_{6'6}) + j^*(\overline{C}_{6'7}) = j^*(C_{67}),$$

where k is any curve on S. Combining the two equations,

$$J^*(\overline{C}_{56'}) + j^*(\overline{C}_{6'7}) - [J^*(\overline{C}_{56}) + j^*(C_{67})]$$
$$= - J^*(k_{6'6}) - j^*(k_{66'}) = J^*(k_{66'}) - j^*(k_{66'}).$$

If expressed in the form of a definite integral, the right hand member of this equation is

$$\int_6^{6'} \{a'F_{x'} + b'F_{y'} + c'F_{z'} - a'f_{x'} - b'f_{y'} - c'f_{z'}\}dt,$$

where a', b', c' are the direction cosines of k. But condition III tells at once that the integrand is identically zero and the above statement follows.

Hence if C_{567} coincides with an extremal for J from 5 to 6 and with an extremal for j from 6 to 7, then

$$J^*(C_{56}) + j^*(C_{67}) = J(C_{56}) + j(C_{67}).$$

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* For a statement of these properties see Bliss and Mason, I, p. 458.

Therefore when $\overline{C}$ is any curve in F' and F'' joining the points 1 and 3 and C is the corresponding curve satisfying conditions I, II, III, IV′, the preceding equations give the relation

$$J^*(\overline{C}_{12'}) + j^*(\overline{C}_{2'3}) = J^*(C_{12}) + j^*(C_{23}) = J(C_{12}) + j(C_{23}).$$

This equation may also be written in the form

$$\begin{aligned} J(\overline{C}_{12'}) + j(\overline{C}_{2'3}) - [J(C_{12}) + j(C_{23})] \\ &= J(\overline{C}_{12'}) - J^*(\overline{C}_{12'}) + j(\overline{C}_{2'3}) - j^*(\overline{C}_{2'3}) \\ &= \int_{\overline{C}_{12'}} Edt + \int_{\overline{C}_{2'3}} edt, \end{aligned}$$

where E and e are the extended Weierstrass E-functions. On account of II these functions are nowhere negative and therefore

$$J(\overline{C}_{12'}) + j(\overline{C}_{2'3}) > J(C_{12}) + j(C_{23}).$$

Hence the conclusion: *Under the hypothesis imposed the curve C_{123} actually minimizes the sum of the integrals J and j if it satisfies the conditions* I, II, III, IV′.

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THE DEGREE OF A CARTESIAN MULTIPLIER.

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1. A large part of Laguerre's numerous and important contributions to the theory of algebraic equations* is based on Descartes' rule of signs, and especially on its application to infinite series. One of the most fertile ideas developed is that an upper limit for the number of real roots of a polynomial with real coefficients, $f(x)$, in an interval $[0, a]$ results from the application of the rule of signs to a product $f_2(x) = f_1(x)f(x)$ developed in a power series which converges for $|x| < a$, but

* See in particular the memoir, "Sur la théorie des équations numériques," Oeuvres, pp. 3–47.