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NOTE ON THE GAMMA FUNCTION.

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IN this note I wish to present a simple development of the principal properties of the function $\Gamma(x)$, based on the elements of the theory of functions of a complex variable.

1. Let $\varphi(x)$ denote the function

$$x^{x-\frac{1}{2}}e^{-x}\sqrt{2\pi} \quad (-\pi < \arg x < +\pi), \tag{1}$$

where that determination of $\varphi(x)$ is chosen which is real and positive when the complex variable x is real and positive. The function $\Gamma(x)$ is defined to be

$$\lim_{n=\infty} \frac{1}{x}\cdot\frac{1}{x+1}\cdots\frac{1}{x+n}\varphi(x+n+1) \tag{2}$$

$$(n = 0, 1, 2, \cdots;\ x \neq 0, -1, -2, \cdots).^{*}$$

It is necessary to establish first that the limit of the sequence exists. Denote the $(n+1)$th term of the sequence by $p_n(x)$. It is clear that none of these terms vanish, and that the question of convergence of the sequence is essentially the same as of the series

$$\log p_0(x) + \log [p_1(x)/p_0(x)] + \log [p_2(x)/p_1(x)] + \cdots, \tag{3}$$

where the principal logarithms are taken.

We have at once the relation

$$\frac{p_n(x)}{p_{n-1}(x)} = \frac{\varphi(x+n+1)}{(x+n)\varphi(x+n)}. \tag{4}$$

* A similar formula has been obtained by Enneper, Dissertation, Göttingen (1856), p. 10.

Let us therefore consider the function $\log[\varphi(x+1)/x\varphi(x)]$, since from it may be obtained the $(n+1)$th term of (3) by replacing x by $x+n$. A substitution of the known value of $\varphi(x)$ gives us at once

$$\log\frac{\varphi(x+1)}{x\varphi(x)} = \left(x+\frac{1}{2}\right)\log\left(1+\frac{1}{x}\right)-1$$
$$= \left(\frac{1}{3}-\frac{1}{2\cdot 2}\right)\frac{1}{x^2} - \left(\frac{1}{4}-\frac{1}{2\cdot 3}\right)\frac{1}{x^3}+\cdots,$$

the last member being the expansion of the function in powers of $1/x$ in the vicinity of $x=\infty$. This expansion converges for $|x|>1$ since the function has no singularity save two branch points of infinite order at 0 and -1; the expansion in series will converge uniformly for $|x| \geqq d > 1$, and the function it represents may be written $M(x)/x^2$, where $M(x)$ remains finite and analytic for $|x| \geqq d$. That is, we have

$$\log\frac{\varphi(x+1)}{x\varphi(x)} = \frac{M(x)}{x^2} \quad (|M(x)| \leqq K \text{ if } |x| \geqq d).$$

The series (3) may now be written

$$(3') \quad \log\varphi(x) + \frac{M(x)}{x^2} + \frac{M(x+1)}{(x+1)^2} + \frac{M(x+2)}{(x+2)^2} + \cdots.$$

Suppose first that x lies in the right half-plane and also that $|x| \geqq d$. In this case $x+1$, $x+2$, $\cdots$ will exceed d in absolute value, and the $(n+1)$th term of (3′) will be not greater in absolute value than the nth term of the series of positive quantities

$$(5) \quad \frac{K}{|x|^2} + \frac{K}{|x+1|^2} + \frac{K}{|x+2|^2} + \cdots.$$

For x in the right half-plane we have $u \geqq 0$, if $x = u + \sqrt{-1}v$, so that

$$|x+n|^2 = (u+n)^2 + v^2 \geqq u^2 + v^2 + n^2 = |x|^2 + n^2.$$

Accordingly each term of the above series is not greater in absolute value than the corresponding term of

$$(6) \quad \frac{K}{|x|^2} + \frac{K}{|x|^2+1^2} + \frac{K}{|x|^2+2^2} + \cdots.$$

If we put $x = d$ in this series we do not decrease the absolute value of any term, and obtain a convergent series of positive constants. Therefore by Weierstrass's test the original series (3) converges absolutely and uniformly for $|x| \geqq d$ and x in the right half-plane. This demonstrates that the limit (2) exists, is analytic, and does not vanish, for x restricted in the manner stated.

By definition of $p_n(x)$ we obtain for $k = 0, 1, \cdots,$

$$(7) \qquad p_n(x) = \frac{1}{x} \cdot \frac{1}{x+1} \cdots \frac{1}{x+k}\, p_{n-k-1}(x+k+1) \qquad (x \neq 0, -1, \cdots).$$

Also for any x we may choose k so large that $x + k + 1$ lies to the right of the imaginary axis, and exceeds d in absolute value. From what precedes we see that the last factor on the right in (7) will uniformly approach the limit $\Gamma(x + k + 1)$ different from zero, in the vicinity of this x, as n becomes infinite. Thus the limit (2) exists in all cases and represents a function $\Gamma(x)$ nowhere zero, analytic in the entire plane with the exception of the points $0, -1, \cdots$. It is clear from the formula (7) that at these points $\Gamma(x)$ has a pole of the first order.

If in particular we put $k = 0$ in (7) and let n become infinite we see that $\Gamma(x)$ is a solution of the functional equation in $f(x)$

$$(8) \qquad f(x+1) = xf(x).$$

The function $\Gamma(x)$ *given by* (2) *is, for* $x \neq 0, -1, -2, \cdots,$ *a single-valued and analytic function different from zero and satisfying the relation* $\Gamma(x+1) = x\Gamma(x)$. *At the excluded points* $\Gamma(x)$ *has a pole of the first order.*

2. It remains to characterize $\Gamma(x)$ in the vicinity of $x = \infty$. For x in the right half-plane and $|x| \geqq d$, we have seen that the $(n + 1)$th term of (3′) is not greater in absolute value than the nth term of (6); this in turn is not greater than the nth term of the series

$$\frac{K}{|x|^2} + K\int_0^1 \frac{dt}{|x|^2 + t^2} + K\int_1^2 \frac{dt}{|x|^2 + t^2} + \cdots,$$

whose sum evaluates to $K/|x|^2 + K\pi/2|x|$. In this way a simple upper limit of the form $K'/2|x|$ for the sum of the terms of (3′) after the first is obtained when $|x| \geqq d$.

For x in the left half-plane and for $|v| \geqq d$, all of the quantities $x+1, x+2, \cdots$, having the same imaginary component, are also at least as great as d in absolute value. Accordingly the $(n+1)$th term of (3′) is again not greater in absolute value than the nth term of (5). In this case the sum of the series (5) is less than $K'/|v|$. In fact, the terms after the leading nth one for which $x+n$ lies on or to the right of the imaginary axis constitute a series of the form before considered of sum less in absolute value than $K/|x+n|^2 + K\pi/2|x+n|$ and therefore less than $K'/2|v|$. The remaining terms constitute a part of a similar series

$$\frac{K}{|-x-n+1|^2} + \frac{K}{|-x-n+2|^2} + \cdots,$$

also less in absolute value than the same quantity.

If then l denotes the distance of the point x from the negative half of the real axis,* we find from (3′) that

$$\log \Gamma(x) = (x - \tfrac{1}{2}) \log x - x + \log \sqrt{2\pi} + \mu(x), \qquad (|\mu(x)| < K'/l \text{ for } l \geqq d.) \tag{9}$$

This result gives at once the following: *the function* $\Gamma(x)$ *has the property that lim* $\Gamma(x)/\varphi(x) = 1$ *if the point* x *recedes indefinitely from the negative half of the real axis.*

3. The fact that $\Gamma(x)$ satisfies the functional equation (8) is of central importance. By means of this fact the function $e^{\pi\sqrt{-1}x}/\Gamma(1-x)$ can also be seen to be a solution of (8). The ratio of $\Gamma(x)$ to this new solution is periodic of period 1, for if x is changed to $x+1$ the equation (8) shows that the first and second members of the ratio are both multiplied by x. This ratio

$$p(x) = \Gamma(x)\Gamma(1-x)/e^{\pi\sqrt{-1}x}$$

may also be verified directly to be of period 1.

Consider now $p(x)$ in the period strip $0 \leqq u \leqq 1$ where it has no singularities save a pole of the first order at $x = 0, 1$; it is clear from the definition of $p(x)$ that this periodic function nowhere vanishes in the strip.

* If $u \leqq 0$ we have $l = |v|$ and if $u \geqq 0$ we have $l = |x|$.

Let us determine the asymptotic form of $p(x)$ at the ends of the strip. We have by (9)

$$\log p(x) = (x - \tfrac{1}{2})[\log x - \log(1 - x)] - 1 + \log 2\pi + \mu(x) + \mu(1 - x) - \pi\sqrt{-1}x.$$

If x lies in the upper half of the strip, $1 - x$ lies in the lower half of the strip, and *vice versa*. Let x tend to infinity in the upper half of the strip, then $1 - x$ will tend to infinity in the lower half. By virtue of our convention we shall have

$$\log(1 - x) = -\pi\sqrt{-1} + \log(x - 1),$$

where $\log(x - 1)$ is the principal logarithm. Substituting, we find

$$\log p(x) = \left[-\left(x - \frac{1}{2}\right)\log\left(1 - \frac{1}{x}\right) - 1\right] - \frac{\pi\sqrt{-1}}{2} + \log 2\pi + \mu(x) + \mu(1 - x).$$

This equation proves that, as x becomes infinite in the upper end of the strip,

$$\lim p(x) = -2\pi\sqrt{-1},$$

since the first term and the last two terms on the right-hand side approach zero. Likewise, as x approaches infinity in the lower half-strip, we have

$$\log(1 - x) = \pi\sqrt{-1} + \log(x - 1)$$

and find

$$\lim e^{2\pi\sqrt{-1}x}p(x) = 2\pi\sqrt{-1}.$$

Accordingly if we write

$$z = e^{2\pi\sqrt{-1}x}, \quad p(x) = q(z),$$

the strip in the x-plane is transformed into the complete z-plane, the upper and lower ends of the strips corresponding to $z = 0$ and $z = \infty$ respectively; and at the same time it is evident that $q(z)$ is single-valued, analytic in the extended z-plane save at $z = 1 (x = 0)$, where it has a pole of the first order; furthermore $q(z)$ vanishes nowhere save at $z = \infty$, where $zq(z)$ takes the value $2\pi\sqrt{-1}$. These facts show at once that $q(z)$ is the rational function of z

$$\frac{2\pi\sqrt{-1}}{(z-1)}.$$

Substituting the corresponding value for $p(x)$ in the equation of definition, there results

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}, \tag{10}$$

a fundamental formula. If both sides are multiplied through by x, the left-hand side may be written $\Gamma(x+1)\Gamma(1-x)$ and tends to $\Gamma^2(1)$ as x tends to zero; the right-hand side tends to 1. Since, by (2), $\Gamma(1)$ is positive we obtain

$$\Gamma(1) = 1. \tag{11}$$

4. According to the results of section 2, we have the important relation

$$\lim_{l=\infty} \frac{\Gamma(x+y)}{x^y\Gamma(x)} = 1, \tag{12}$$

where y is fixed and l again represents the distance of the point x from the nearest point of the negative half of the axis of reals; in fact these results show that this limit is equal to

$$\lim_{l=\infty} \frac{\varphi(x+y)}{x^y\varphi(x)} = \lim_{l=\infty}\left(1+\frac{y}{x}\right)^{x+y-\frac{1}{2}} e^{-y} = 1. \tag{13}$$

Now if we divide $p_n(x)$ by $p_{n-1}(1)$ we find at once

$$\frac{p_n(x)}{p_{n-1}(1)} = \frac{1\cdot 2\cdots n}{x\cdot x+1\cdots x+n}\,\frac{\phi(x+n+1)}{\phi(n+1)};$$

if further we make use of equations (11) and (13), letting n become infinite, we obtain Euler's formula

$$\Gamma(x) = \lim_{n=\infty} \frac{1\cdot 2\cdots n}{x\cdot x+1\cdots x+n}(n+1)^x. \tag{14}$$

It is to be noted that the final factor on the right, which replaces $\varphi(x+n+1)/\varphi(n+1)$ in $p_n(x)/p_{n-1}(1)$, is not of such a nature as to affect the uniform convergence of the logarithm of the sequence. This property is carried over from (3) since the final factor bears to the factor which it replaces a ratio which approaches 1 uniformly in any finite region as n becomes infinite.

If we introduce Euler's constant

$$C = \lim_{n=\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right),$$

we obtain the Schlömilch product formula

$$\Gamma(x) = \frac{e^{-Cx}}{x} \prod_{1}^{\infty} \frac{e^{x/n}}{1 + x/n}. \tag{15}$$

In this case the final factor on the right in (14) is replaced by

$$e^{\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - C\right)x}.$$

5. Another form of solution of the functional equation is the definite integral

$$\int_0^{+\infty} e^{-t} t^{x-1} dt,$$

valid for x in the right half-plane. The fact that this integral $I(x)$ satisfies the functional equation may be verified by forming $xI(x)$ and integrating by parts. It is furthermore evident that $I(1) = 1$.

To prove that $I(x)$ is identical with $\Gamma(x)$ we can proceed as follows. The function $I(x)$ is analytic for any x within the right half-plane, since the integrand is analytic in x for $t > 0$; and the integral is absolutely and uniformly convergent in the vicinity by the ordinary tests. The ratio function

$$p(x) = I(x)/\Gamma(x)$$

is periodic of period 1 in x in consequence of the fact that $I(x)$ and $\Gamma(x)$ are solutions of (8); and this function $p(x)$ is analytic in x throughout any period strip such as $1 \leqq u \leqq 2$.

Now the path of integration along the positive half of the real axis may be modified to be any ray within the right half-plane from $t = 0$ to $t = \infty$. In fact the integrand is continuous in the sector formed by the positive half of the real axis and such a ray, and vanishes to infinite order at $t = \infty$; thus Cauchy's integral theorem may be applied to show that the integral taken around the sector vanishes. Take the ray to pass through $t = x$ in the selected period strip and write $t = x\rho$; we obtain

$$I(x) = x^x \int_0^{+\infty} e^{-x\rho}\rho^{x-1}d\rho,$$

where ρ is a real variable. Hence we see that for $1 \leqq w \leqq 2$

$$| I(x) | \leqq | x^x | \left\{ \int_0^1 d\rho + \int_1^\infty e^{-\rho}\rho d\rho \right\} < 2 | x^x |,$$

since if the second integral in the bracketed expression were taken from 0 to ∞ it would give $I(2) = I(1) = 1$ by (8) and (11). Moreover by (9) it is clear that

$$| \Gamma(x) > | x^{x-1} | \quad \text{for} \quad 1 \leqq u \leqq 2, \quad | v | > \lambda,$$

if λ is sufficiently large and positive.

Consequently we obtain, for $| v |$ sufficiently large,

$$| p(x) | < 2 | x |.$$

As before, write

$$z = e^{2\pi\sqrt{-1}x}, \quad p(x) = q(z).$$

It is apparent that $q^{(z)}$ is single-valued and analytic at every point save $z = 0$ and $z = \infty$, where however $zq(z)$ and $q(z)/z$ respectively tend to zero. It follows by Riemann's theorem that $q(z)$ is analytic in the extended plane, and thus is a constant which reduces to 1 since it has already been seen that $I(1) = \Gamma(1) = 1$.

6. Differentiating Schlömilch's infinite product logarithmically, we obtain the series for the Euler ψ-function

$$(16) \quad \psi(x) = \frac{d}{dx}\log\Gamma(x) = -C - \frac{1}{x} - \sum_1^\infty \left[\frac{1}{x+n} - \frac{1}{n}\right].$$

The differentiation is legitimate because of the uniformity of convergence of this product. From the properties of $\Gamma(x)$ it follows that $\psi(x)$ *is analytic in the finite plane save for poles of the first order at* $0, -1, \ldots$ *with residues* -1 *at these points.*

We furthermore obtain

$$(17) \quad \psi(x+1) - \psi(x) = \frac{1}{x}, \quad \psi(x) - \psi(1-x) = \pi\cot\pi x$$

directly from the functional equation for $\Gamma(x)$ and from (10).

To investigate the nature of $\psi(x)$ at $x = \infty$, we differentiate (9), which then becomes

$$\psi(x) = \log x - \frac{1}{2x} + \frac{d}{dx}\mu(x),$$

and consider the magnitude of the last term. Take a point x at a distance $l \geqq 2d$ from the negative half of the real axis, and about it draw a circle C of radius $l/2$. Then we have by differentiating Cauchy's integral for $\mu(x)$, and using the upper limit for $\mu(x)$ given in (9),

$$\left|\frac{d}{dx}\mu(x)\right| = \left|\frac{-1}{2\pi\sqrt{-1}}\int_C \frac{\mu(t)}{(t-x)^2}dt\right| \leqq \frac{2K'}{\pi l^3}\int_C |\,dt\,| = \frac{2K'}{l^2}.$$

This desired inequality shows that $d\mu/dx$ is of the second order in $1/l$ and that *lim* $\psi(x)/\log x = 1$ *as the distance of the point* x *from the negative half of the real axis becomes infinite in any manner.*

7. A final central theorem is the development of the function $\Gamma(x)$ relates to the evaluation of the beta function

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \tag{18}$$

where x and y have positive real parts. The formula to be proven is

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{19}$$

The attack is identical in spirit with that followed in section 5.

First we observe that $B(x, y)$ is analytic and symmetric in x, y in the domain under discussion and that $B(1, 1) = 1$. Also, integrating by parts, we find

$$xB(x, y+1) = yB(x+1, y),$$

and we find directly

$$B(x, y+1) = \int_0^1 t^{x-1}(1-t)^{y-1}d - \int_0^1 t^{x}(1-t)^{y-1}dt$$
$$= B(x, y) - B\,(x+1, y).$$

Thus we obtain the functional equation in x

$$B(x+1, y) = \frac{xB(x, y)}{x+y},$$

and likewise the same functional equation in y. It is readily verified that the right-hand member of (19) satisfies these equations and hence that

$$p(x, y) = \frac{B(x, y)\Gamma(x+y)}{\Gamma(x)\Gamma(y)}$$

is a function periodic in x and y of period 1 and analytic in the domain under consideration.

Now hold y fixed of real part not less than 1, and let x be arbitrary in the period strip $1 \leqq u \leqq 2$. We have then

$$| B(x, y) | \leqq \int_0^1 | t^{x-1}(1-t)^{y-1} | \, dt \leqq 1.$$

Moreover we have for some $k > 0$

$$\left|\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\right| > k \left|\frac{\Gamma(x)}{\Gamma(x+y)}\right| > \frac{k}{2} | x^{-y} |$$

by (12) for $| v |$ large enough. Thus there results

$$| p(x, y) | < \frac{2|x^y|}{k}$$

for $| v |$ large enough. Since $p(x, y)$ is analytic throughout the finite strip in x, and since this relation shows that $p(x, y)$ is finite at both ends of the strip (see section 5), it follows that $p(x, y)$ is finite throughout the strip. Hence $p(x, y)$ is constant in x, and likewise in y, and therefore is constant in both variables, being equal to $p(1, 1) = 1$. This demonstrates the truth of (19).

The general type of consideration given above admits of further development, but the material which I have presented will serve to indicate its nature.

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