

THE APRIL MEETING OF THE AMERICAN
MATHEMATICAL SOCIETY.

THE one hundred and sixty-third regular meeting of the Society was held in New York City on Saturday, April 26, 1913. The attendance at the two sessions included the following fifty-seven members:

Professor R. C. Archibald, Professor M. J. Babb, Dr. F. W. Beal, Mr. R. D. Beetle, Mr. A. A. Bennett, Professor W. J. Berry, Professor E. G. Bill, Professor G. D. Birkhoff, Dr. Henry Blumberg, Professor Maxime Bôcher, Professor B. H. Camp, Professor C. W. Cobb, Dr. Emily Coddington, Professor F. N. Cole, Dr. G. M. Conwell, Professor J. L. Coolidge, Mr. C. H. Currier, Dr. L. L. Dines, Professor L. P. Eisenhart, Professor T. S. Fiske, Professor W. B. Fite, Mr. Meyer Gaba, Mr. G. M. Green, Professor C. C. Grove, Professor H. E. Hawkes, Professor E. V. Huntington, Dr. W. A. Hurwitz, Dr. Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Dr. S. D. Killam, Mr. E. H. Koch, Jr., Dr. D. D. Leib, Professor W. R. Longley, Professor J. H. Maclagan-Wedderburn, Dr. H. H. Mitchell, Dr. R. L. Moore, Dr. F. M. Morgan, Mrs. Anna J. Pell, Professor A. D. Pitcher, Dr. H. W. Reddick, Professor R. G. D. Richardson, Dr. J. E. Rowe, Professor L. P. Siceloff, Mr. C. G. Simpson, Mr. L. L. Smail, Professor D. E. Smith, Professor P. F. Smith, Professor H. D. Thompson, Mr. H. S. Vandiver, Mr. C. E. Van Orstrand, Professor E. B. Van Vleck, Professor Oswald Veblen, Mr. H. E. Webb, Professor H. S. White, Mr. K. P. Williams, Professor J. W. Young.

The President of the Society, Professor E. B. Van Vleck, occupied the chair, being relieved by Ex-Presidents White and Fiske. The Council announced the election of the following persons to membership in the Society: Professor E. H. Jones, Daniel Baker College; Mr. L. B. Robinson, Johns Hopkins University; Dr. H. M. Sheffer, Cornell University; Dr. W. B. Stone, University of Michigan; Professor F. B. Wiley, Denison University. Six applications for membership in the Society were received.

Professor D. R. Curtiss was appointed a member of the Editorial Committee of the *Transactions*, to succeed Professor

Bôcher, whose term expires in October, 1913. Professor P. F. Smith was appointed to succeed Professor White, who desires to retire from the *Transactions* Committee at the close of the present year.

As already announced, the summer meeting and colloquium of the Society will be held at the University of Wisconsin during the week September 8-13. It was decided to hold the annual meeting this year in New York City on Tuesday and Wednesday, December 30-31.

The following papers were read at the April meeting.

(1) Dr. E. L. DODD: "The error risk of the median compared with that of the arithmetic mean."

(2) Mr. P. M. BATCHELDER: "The divergent series satisfying linear difference equations of the second order."

(3) Mr. P. M. BATCHELDER: "The hypergeometric difference equation."

(4) Mr. H. J. ETLINGER: "On a generalization of a Sturmian boundary problem."

(5) Dr. R. L. MOORE: "Concerning pseudo-Archimedean and Vollständigkeit axioms."

(6) Dr. J. E. ROWE: "The relation between the pencil of tangents from a point to a rational plane curve and their parameters."

(7) Professor E. G. BILL: "Analytic curves in non-euclidean space (third paper)."

(8) Mr. JOSEPH SLEPIAN: "On the functions of a complex variable defined by a differential equation of the first order and the first degree."

(9) Dr. NATHAN ALTSHILLER: "On the cubic with a double point."

(10) Dr. C. F. CRAIG: "Ruled surfaces associated with certain rational space curves."

(11) Dr. H. M. SHEFFER: "The generalized principle of duality in Boolean algebras."

(12) Dr. T. H. GRONWALL: "On the maximum modulus of an analytic function."

(13) Mr. L. L. SMAIL: "Note on the summability of properly divergent series."

(14) Dr. MAURICE FRÉCHET: "Sur les classes V normales."

(15) Professor MAXIME BÔCHER: "An application of the conception of adjoint systems."

(16) Professor G. D. BIRKHOFF: "Note on the gamma function."

(17) Professor G. D. BIRKHOFF: "Solution of the generalized Riemann problem for linear differential equations, and of the analogous problem for linear difference and q -difference equations."

(18) Professor L. P. EISENHART: "Transformations of surfaces of Guichard."

(19) Professor E. V. HUNTINGTON: "Sets of independent postulates for betweenness (second paper)."

(20) Professor A. D. PITCHER: "On the connection of an abstract set, with applications to the theory of functions of a general variable."

(21) Professor A. D. PITCHER: "Concerning the property Δ of a class of functions."

(22) Professor R. G. D. RICHARDSON: "Oscillation theorems for a system of n linear self-adjoint partial differential equations of the second order with n parameters."

(23) Dr. H. H. MITCHELL: "On some systems of collineation groups."

(24) Mr. H. S. VANDIVER: "Symmetric functions formed by certain systems of elements of a finite algebra, and their connection with Fermat's quotient and Bernoulli's numbers."

(25) Dr. C. A. FISCHER: "The derivative of a function of a surface."

(26) Dr. C. T. SULLIVAN: "Properties of surfaces whose asymptotic curves belong to linear complexes."

(27) Dr. S. D. KILLAM: "A note on graphical integration of functions of a complex variable."

(28) Mr. K. P. WILLIAMS: "On the asymptotic form of the function $\Psi(x)$."

(29) Mr. M. G. GABA: "A set of postulates for general projective geometry in terms of point and transformation."

(30) Dr. W. A. HURWITZ: "Postulate sets for abelian groups and fields."

(31) Professor EDWARD KASNER: "The interpretation of the Appell transformation."

(32) Mr. G. M. GREEN: "Systems of k -spreads in an n -space."

(33) Professor J. W. YOUNG: "A new formulation for general algebra."

(34) Professor J. W. YOUNG and Dr. F. M. MORGAN: "The geometry associated with a certain group of cubic transformations in space."

Mr. Slepian was introduced by Professor Birkhoff, Dr. Altshiller and Dr. Fischer by the Secretary. The papers of Mr. Batchelder and Mr. Ettlinger were communicated to the Society by Professor Birkhoff, and that of Dr. Fréchet by Professor Bôcher. The papers of Dr. Dodd, Mr. Batchelder, Mr. Ettlinger, Dr. Craig, Dr. Sheffer, Dr. Gronwall, Mr. Smail, Dr. Fréchet, Dr. Sullivan, Mr. Williams, Professor Kasner, Mr. Green, Professor Young, and Professor Young and Dr. Morgan were read by title.

The second paper of Professor Pitcher and the paper of Mr. Williams appeared in full in the June BULLETIN. Dr. Killam's paper is published in the present number of the BULLETIN. Dr. Fréchet's paper will appear in the *Transactions*.

Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. It is not at present known what function of the measurements the Gaussian probability law endorses above all other functions; and perhaps there is no such function. The arithmetic mean M of n measurements can not claim this distinction;* indeed, the error risk of $(1 - 1/n^2)M$ is less than that of M when n is large enough.†

Dr. Dodd shows, however, that under the Gaussian law the error risk of the arithmetic mean is less than that of the median.

The median is a function of the measurements which lacks certain partial derivatives at some points, and it can not be given a Taylor development about a point with all coordinates equal. Thus, it is not one of the functions considered by Czuber in his treatment of error risk, "Fehlerrisiko."‡ The median, however, has found considerable favor among biologists and economists; and in certain non-Gaussian distributions is probably superior to the arithmetic mean.

2. The general existence theorem for the theory of linear difference equations, as worked out by Birkhoff, is based on the formal solutions of the equations in terms of divergent series. These series are known to exist only when all the roots of the characteristic equation are distinct from each

* See Bertrand, *Calcul des Probabilités* (1889), p. 180.

† This BULLETIN, February, 1913, p. 223, inequality (1).

‡ *Wahrscheinlichkeitsrechnung*, I, p. 276.

other and from zero. The methods of Nörlund and Galbrun, based on the Laplace transformation, lead to great difficulties in the cases of equal or zero roots, while the method of Birkhoff can be applied readily provided a complete set of divergent series has been found. In the present paper Mr. Batchelder proves the existence of formal series for equations of the second order in all possible cases. The difficulties of extending these results to equations of the n th order are chiefly algebraical, and the author hopes shortly to present a complete discussion of the question.

3. In this paper Mr. Batchelder makes a thorough study of the properties of difference equations of the form

$$(a_2x + b_2)f(x + 2) + (a_1x + b_1)f(x + 1) + (a_0x + b_0)f(x) = 0,$$

including all exceptional cases. Assuming first that the characteristic equation possesses two distinct finite roots different from zero, analytic solutions are obtained in the well-known form of definite integrals by means of the Laplace transformation, and evaluated (with Barnes) in terms of hypergeometric series and gamma functions.

The equation is next shown to admit of formal solution by $S_1(x)$ and $S_2(x)$, two divergent power series in $1/x$ multiplied by exponential factors. Four of the definite integral solutions are characterized by their asymptotic forms as $|x|$ approaches ∞ , namely, $g_1(x) \sim S_1(x)$ and $g_2(x) \sim S_2(x)$ in the left half plane, and $h_1(x) \sim S_1(x)$ and $h_2(x) \sim S_2(x)$ in the right half plane; these are the principal solutions. The periodic functions by which $g_1(x)$ and $g_2(x)$ are expressed in terms of $h_1(x)$ and $h_2(x)$ are then determined explicitly, and with their aid the asymptotic forms of the principal solutions are studied in the complete vicinity of ∞ . The so-called intermediate solutions are obtained in explicit form, and the properties of the remaining definite integral solutions are studied. The fundamental properties of the solutions of the equation having been thus determined, other questions are considered, such as the conditions for reducibility, etc.

One of the main objects of the paper is reached in the extension of the general theory to the irregular cases in which the roots of the characteristic equation are equal to each other or equal to 0 or ∞ . The principal solutions are derived, and the formal solutions which represent them asymptotically for large

values of $|x|$ are obtained in all cases. One interesting fact discovered is that some of the series proceed according to powers of $1/\sqrt{x}$, as in the anormal series for differential equations.

4. In Mr. Ettlenger's paper the following boundary-value problem is considered:

$$\begin{aligned} \frac{d}{dx} \left[K(x, \lambda) \frac{du}{dx} \right] - G(x, \lambda)u &= 0, \\ \alpha_0(\lambda)u(a, \lambda) + \beta_0(\lambda)K(a, \lambda)u'(a, \lambda) \\ &= \gamma_0(\lambda)u(b, \lambda) + \delta_0(\lambda)K(b, \lambda)u'(b, \lambda), \\ \alpha_1(\lambda)u(a, \lambda) + \beta_1(\lambda)K(a, \lambda)u'(a, \lambda) \\ &= \gamma_1(\lambda)u(b, \lambda) + \delta_1(\lambda)K(b, \lambda)u'(b, \lambda), \\ \alpha_0\beta_1 - \beta_0\alpha_1 &= \gamma_0\delta_1 - \delta_0\gamma_1. \end{aligned}$$

The existence of an infinite number of characteristic values for this system is proved under the restriction that K and G decrease as λ increases. Mr. Ettlenger makes use of a striking symmetry which is exhibited by the above equation when we set $z = Ku'$, $u' = Hz$, $z' = Gu$, where $H = 1/K$. The method is that used by Bôcher and Birkhoff in treating similar boundary-value problems, and is based on Sturm's fundamental theorems of oscillation. The proof of this theorem rests on the fact that between every pair of characteristic numbers of the associated Sturmian system

$$\begin{aligned} \alpha_0(\lambda)u(a, \lambda) + \beta_0(\lambda)K(a, \lambda)u'(a, \lambda) &= 0, \\ \gamma_0(\lambda)u(b, \lambda) + \delta_0(\lambda)K(b, \lambda)u'(b, \lambda) &= 0 \end{aligned}$$

there is to be found at least one characteristic number of the given problem. If l_1, l_2, \dots be these characteristic numbers and $\lambda_1, \lambda_2, \dots$ those of the Sturmian system, in general two cases arise, according as $\lambda_{2p-1} \leq l_{2p} \leq \lambda_{2p}$ or $\lambda_{2p} \leq l_{2p} \leq \lambda_{2p+1}$. Furthermore, if

$$\begin{aligned} (\alpha_0'\beta_0 - \alpha_0\beta_0') - (\gamma_0'\delta_0 - \gamma_0\delta_0') &\geq 0, \\ (\alpha_1'\beta_1 - \alpha_1\beta_1') - (\gamma_1'\delta_1 - \gamma_1\delta_1') &\geq 0, \\ (\alpha_0\beta_1' - \alpha_1'\beta_0) - (\gamma_1\delta_0' - \gamma_0'\delta_1) &= 0, \end{aligned}$$

there is only one value in each such interval. The means for discriminating between these cases is found in the appendix. An oscillation theorem follows for the roots of the solution $u_{p+1}(x)$ which corresponds to $\lambda = l_{p+1}$.

The boundary conditions are then put into a normal form $L[u(a, \lambda)] = k(\lambda)L[u(b, \lambda)]$, $M[u(a, \lambda)] = 1/k(\lambda)M[u(b, \lambda)]$,

where

$$L[u(x, \lambda)] = \alpha(\lambda)u(x, \lambda) + \beta(\lambda)K(x, \lambda)u'(x, \lambda),$$

$$M[u(x, \lambda)] = \gamma(\lambda)u(x, \lambda) + \delta(\lambda)K(x, \lambda)u'(x, \lambda),$$

and k may be real or complex with modulus 1. For the case k real, a complete oscillation theorem is obtained for the roots of $L[u_{p+1}(x)]$ and $M[u_{p+1}(x)]$. The special case $k = 1$ requires consideration, and the resulting theorem yields two possibilities for the roots of $M[u_{p+1}(x)]$. For k complex, the oscillation theorem gives no specific information.

5. Consider the following axioms:

Axiom A . If (a) the set of all points of a line l lying in a plane M be divided into two subsets S_1 and S_2 such that no point of S_1 is between two points of S_2 ; (b) A and B are two points lying in M on the same side of l ; (c) the line AB does not separate every point of S_1 (not on AB) from every point of S_2 (not on AB); (d) P_1 is a point of S_1 and P_2 is a point of S_2 and C is a point lying in the plane M on the opposite side of l from A and B ; then there exists a point X within the triangle P_1CP_2 such that the triangle AXB contains at least one point of S_1 and at least one point of S_2 .

Axiom V . If the set of all points in a plane M be divided into two mutually exclusive subsets S_1 and S_2 , then there exists a point X in one of these subsets such that every triangle containing X and lying in M contains points belonging to the other subset.

Let H_1 denote Hilbert's plane axioms of groups I and II,* or Veblen's Axioms I-VIII.† Let H_2 denote H_1 together with Desargues's theorem (considered as an axiom) and, say, Hilbert's Axiom III of parallels.

Dr. Moore shows that

* Grundlagen der Geometrie.

† Transactions, vol. 5 (1904), No. 3, pp. 344 and 345.

I. If a plane M satisfies H_2 together with Axiom A , then M is an ordinary euclidean plane with possibly certain points omitted.

II. If a plane M satisfies H_1 and Axiom A , then, though M may not be either a euclidean nor a Lobachevskian plane with respect to the "line" of Hilbert's axioms or of Veblen's definitions, nevertheless there exists in M a set of curves such that if they are taken as the lines of M then M will be an ordinary euclidean plane (with perhaps certain points omitted).

III. If a plane M satisfies H_2 , Axiom A , and Axiom V , then it is a categorical euclidean plane.

It seems at least strongly probable however that a plane satisfying H_2 and V is not necessarily a categorical euclidean plane, though a plane satisfying H_2 and "the Dedekind cut postulate" (for every line of that plane) would be a categorical euclidean plane. Thus there seems to be a strong analogy between the Axiom V and Hilbert's axiom of completeness. Clearly Axiom A has some of the properties of an Archimedean axiom.

6. In Dr. Rowe's paper the problem of discovering the relation between the pencil of tangents from a point to a rational plane curve and their parameters is attacked as a purely algebraic problem. The method may be divided into three parts: (1) the derivation of a complete system of covariant curves of the R^n defined by projective relations connecting the pencil of tangents from a point to the R^n , and this requires the further development of a certain type of combinant of two binary forms; (2) the derivation of a complete system of covariant curves of the R^n defined by projective relations connecting the parameters of the tangents from a point to the R^n ; (3) a comparison of curves (1) and (2) is carried out in detail for the R^3 and R^4 , which reveals the fact that no curve of (1) is expressible in terms of the curves (2) alone, and reasons are cited to support the belief that the same sort of relation exists in the case of rational curves of higher order.

In the case of R^3 we find a line, a cubic, and a sextic each of which possesses the property that from any point of it the pencil of tangents to the R^3 and their parameters are projectively equivalent. In case of R^4 besides the curves from which the pencil of tangents and their parameters are projectively equiva-

lent for a certain set of invariants, we find that from any point of the conic $B = 0$ (in the notation of previous papers) a given invariant relation upon the pencil of tangents to the R^4 is equivalent to some other invariant relation upon their parameters, which relation may easily be found by the method described.

7. This paper is a continuation of the work presented at two meetings of the Chicago Section in which Professor Bill applies to the problems of the differential geometry of non-euclidean space the methods of attack used by Study in discussing similar problems in euclidean space.

8. Mr. Slepian considers the differential equation $dy/dx = P(x, y)/Q(x, y)$, where P and Q are polynomials in the complex variables x and y . The integrals of such an equation are infinitely many valued functions over the x plane, the singularities (branch points and poles) apart from the fixed singularities having a certain uniform distribution. In order that all the fixed singularities of the equation may be of the Briot and Bouquet type, it is necessary that the degree of Q in x be at least two greater than the degree of P in x . With this condition satisfied the following is found to be true:

If the equation admits an algebraic integral, then each integral passes through some fixed singularity of the equation. From this it readily follows that if the degree of P in y does not exceed the degree of Q in y by as much as two, then each integral of the equation passes through some fixed singularity. If each integral of the equation passes through some fixed singularity, then each pair of integrals must pass through a common fixed singularity.

For a value x , not the x of a fixed singularity, let an integral $Y(x)$ have in its different branches the values y_1, y_2, \dots ; let z_1, z_2, \dots be limit values of y_1, y_2, \dots ; then the integrals which at the point x take the values z_1, z_2, \dots , are called limit solutions of $Y(x)$. If an integral does not pass through a particular fixed singularity, then none of its limit solutions can pass through that fixed singularity.

Any integral of the equation has among its limit solutions either an algebraic integral, or a system of integrals L , having these properties: (1) any limit solution of an integral of the system L is again an integral of the system L ; (2) each

integral of the system L is a limit solution of any other integral of the system L . No two different L systems can have an integral in common.

In the neighborhood of a fixed singularity, as is well known, the integrals are given by $f(x, y)^\lambda/g(x, y)^\mu = C$. If the ratio of λ to μ is not real, then the integrals $f(x, y) = 0$ and $g(x, y) = 0$ are limit solutions of every integral which passes through the fixed singularity. When the ratio of λ to μ is real for none of the fixed singularities of the equation, then there cannot be more than one L system for the differential equation. If, however, for any of the fixed singularities, the ratio of λ to μ is real, then either there are infinitely many L systems or all the integrals of the equation belong to a single L system; in both cases, each integral of the equation passes through some fixed singularity.

If the equation has any integral which passes through no fixed singularity, then there is one and only one L system. Each integral of this system passes through no fixed singularity. The integrals of this system are remarkably like the algebraic functions in their properties.

9. Five given coplanar points A, A', B, B', O are projected from a point M by the involution of rays $M(OO, AA', BB')$. Dr. Altshiller proves synthetically that the locus of M is a cubic with a double point at O , the couples A, A' and B, B' being couples of corresponding poles on the curve. The construction of the curve leads to several interpretations of the conic ω that is enveloped by the lines joining the couples of corresponding poles on the cubic. The tangent to the cubic at any point L , the two tangents from L to ω , and the line LO form an harmonic set. With respect to a point pencil of conics having for its base any two couples of corresponding poles of the cubic, the conjugate of O is on the inflexional line o . Other relations of the points of o to the couples of corresponding poles on the cubic and the conic ω are considered. Incidentally there comes to light a relation between the line pencil of conics determined by a complete quadrilateral and the three point pencils of conics determined by three couples of opposite vertices of the quadrilateral, when taken in pairs.

10. On a rational space curve the lines joining corresponding

points of any projectivity define a rational scroll having the given curve as a component of its nodal curve. In Dr. Craig's paper conditions for coplanar projectivities are determined. The paper includes the rational separability of the scrolls generated by joining a point P to the residual $n - 3$ points of intersection of the curve and the osculating plane at P , and a number of other special cases.

11. In this paper Dr. Sheffer proves a principle of duality which holds in Boolean algebras—"the algebra of logic"—when these are determined by means of any set of relations or operations. The well-known duality in terms of $+$, \times , 0 , 1 , and inclusion is shown to be a special case. From the theorem, proved in the paper, that Boolean algebras having more than one element cannot be determined by any postulate set involving self-dual relations alone, it follows that Kempe's "base system" (*Proceedings of the London Mathematical Society*, volume 21, 1890) and Royce's "system Σ " (*Transactions*, volume 6, 1905), supposed by their respective authors to determine Boolean algebras, do so only when the system contains but one element.

12. In this paper, Dr. Gronwall considers the maximum modulus of a power series with given initial coefficients and gives quite elementary proofs of certain theorems obtained by Carathéodory and Fejér (*Palermo Rendiconti*, volume 32, 1911) from considerations belonging to Minkowski's geometry of convex solids.

13. In this note, Mr. Smail gives a very general definition of summability, which includes as special cases the well-known definitions of Cesàro, Riesz, LeRoy, Borel's integral definition, and the definitions of the so-called Cesàro-Riesz methods of Hardy and Chapman, and of Euler's power series method. It is proved in a very simple manner that no properly divergent series can be summable according to this general definition with finite generalized sum, and hence it follows at once that none of the particular methods enumerated above will give a finite generalized sum for such a series.

15. A system consisting of an ordinary homogeneous linear differential equation $L(u) = 0$ of the n th order and n homo-

geneous linear boundary conditions $U_i(u) = 0$ at a and b has, as was first pointed out in a general manner by Birkhoff (*Transactions*, volume 9 (1908), page 373), an adjoint system $M(v) = 0$, $V_i(v) = 0$ of the same character, which is such that Green's theorem may be written

$$\int_a^b [vL(u) - uM(v)]dx = U_1V_{2n} + U_2V_{2n-1} + \dots + U_{2n}V_1,$$

where the U 's and V 's with subscripts greater than n are expressions of the same sort as those previously introduced with subscripts less than or equal to n . By applications of Green's theorem in this form it may be proved, on the one hand, that the two homogeneous linear systems just mentioned always have the same number of linearly independent solutions; and on the other, that a necessary and sufficient condition that the general linear system

$$L(u) = r, \quad U_1(u) = \gamma_1, \dots, U_n(u) = \gamma_n$$

have a solution is that every solution of the adjoint system

$$M(v) = 0, \quad V_1(v) = 0, \dots, V_n(v) = 0$$

satisfy the relation

$$\int_a^b vrdx = \gamma_1V_{2n}(v) + \gamma_2V_{2n-1}(v) + \dots + \gamma_nV_{n+1}(v).$$

These are the main results of Professor Bôcher's paper. It is shown how the latter includes as a special case a similar result of Mason (*Transactions*, volume 7 (1906), page 337) for the self-adjoint system of the second order; and also how they admit of extension to systems of linear differential equations of any order.

16. Professor Birkhoff's note contains a proof of some well known formulas; it will appear in the BULLETIN.

17. In a paper published in 1909 (*Transactions*, volume 10, pages 436-470), Professor Birkhoff formulated a generalization of the Riemann problem for ordinary linear differential equations, to include the case of irregular singular points; such an extension was first possible after the theorems of this paper.

Later (*Transactions*, volume 12 (1911), pages 242–284) he formulated the problem for linear difference equations which holds, in its field, a position analogous to that of the Riemann problem in the field of linear differential equations.

The solution of these two problems is effected in the present paper by a direct method of successive approximations. The analogous problem for linear q -difference equations is also formulated, and solved by the same means.

18. When a surface S has associated with it a surface \bar{S} such that the two surfaces have the same spherical representation of their lines of curvature, and the principal radii of curvature $\rho_1, \rho_2; \bar{\rho}_1, \bar{\rho}_2$ of the respective surfaces are in the relation $\rho_1\bar{\rho}_2 + \rho_2\bar{\rho}_1 = \text{const.} \neq 0$, then each surface is said to be a surface of Guichard and the other is its associate; these surfaces are named for the distinguished geometer who first considered them. The determination of surfaces of Guichard requires the solution of two partial differential equations, which are of the third order in one dependent variable and of the first order in the other, as Calapso has shown.

Professor Eisenhart has shown that there exist pairs of surfaces of Guichard which constitute the nappes of the envelope of a two-parameter family of spheres on which the lines of curvature correspond. In this sense either surface may be obtained from the other by a transformation of Ribaucour. The determination of the transforms of a surface S requires the solution of an illimitably integrable system of partial differential equations of the first order involving a parameter. When one knows a transformation of S , a transformation of the associate \bar{S} also is known. There is a subclass of surfaces of Guichard, satisfying an additional differential condition involving four arbitrary constants, each of which possesses the property that it admits a transform S_1 of the same kind, involving the same constants, such that the plane determined by the normals to S and S_1 at corresponding points envelopes a surface applicable to a general quadric. Hence the transformations of certain surfaces of Guichard and the deformation of quadrics are allied problems. When S is of the special class referred to above, so also is \bar{S} , but the constants are interchanged. With each surface of Guichard there are associated ∞^1 isothermic surfaces. The paper

considers also the transformations of these isothermic surfaces arising from the transformations of the surfaces of Guichard.

19. Professor Huntington's paper is a continuation of the paper presented at the December meeting, and presents six different sets of postulates for betweenness. These six sets are selected from the following list of ten propositions, $A-D$, 1-6, in which the notation AXB may be read: " X is between A and B ." (A) If AXB , then BXA . (B) If A , B , and C are distinct, then at least one of the three relations ABC , BCA , CAB is true. (C) If AXY and AYX , then $X = Y$. (D) If ABC , then A , B , and C are distinct. In the remaining postulates, 1-6, the elements A , B , X , Y are supposed to be distinct. (1) If XAB and ABY , then XAY . (2) If XAB and AYB , then XAY . (3) If XAB and AYB , then XYB . (4) If AXB and AYB , then either AXY or AYX . (5) If XAB and YAB , then either XYA or YXA . (6) If XAB and YAB , then either XYB or YXB . These ten propositions include all the possible formal laws of betweenness concerning not more than four distinct elements. Postulates $A-D$ are absolutely independent, and are common to all the sets. Postulates 1-6 contain redundancies, but all the possible ways in which any one of these postulates can be deduced from any other postulates of the list are explicitly set forth in twenty-two theorems. The following six sets are then shown to be the only possible sets of independent postulates that can be selected from the basic list. I. $A-D$, 1, 2. II. $A-D$, 1, 5. III. $A-D$, 1, 6. IV. $A-D$, 2, 4. V. $A-D$, 3, 4, 5. VI. $A-D$, 3, 4, 6. The paper contains twenty-one examples of pseudo-betweenness relations, which are used in the proofs of independence.

20. In this paper Professor Pitcher defines what he calls the connection of an abstract set. By use of this definition he makes a contribution to the theory of functions of a general variable. He secures certain necessary and sufficient conditions which tend to show that the notion connection, as here defined, is fundamental. The paper will appear in the *American Journal of Mathematics*.

22. The chief theorem which Professor Richardson enunciated is general in its character, but was stated for the case of two independent variables and three equations.

Given three two-dimensional regions on the boundary of which the functions $u_1(x_1, y_1)$, $u_2(x_2, y_2)$, $u_3(x_3, y_3)$ shall vanish respectively, and given the equations

$$\frac{\partial}{\partial x_i} \left(p_i \frac{\partial u_i}{\partial x_i} \right) + \frac{\partial}{\partial y_i} \left(p_i \frac{\partial u_i}{\partial y_i} \right) + q_i u_i \\ + (\lambda A_{i1} + \mu A_{i2} + \nu A_{i3}) u_i = 0 \quad (i = 1, 2, 3),$$

where $p_i > 0$, q_i , A_{ij} are functions of x_i , y_i ; in order that solutions u_1 , u_2 , u_3 exist which oscillate n_1 , n_2 , n_3 times respectively, it is necessary that sets of constants $c_1^{(1)}$, $c_1^{(2)}$, $c_1^{(3)}$; $c_2^{(1)}$, $c_2^{(2)}$, $c_2^{(3)}$; $c_3^{(1)}$, $c_3^{(2)}$, $c_3^{(3)}$ exist such that

$$c_1^{(i)} A_{i1} + c_2^{(i)} A_{i2} + c_3^{(i)} A_{i3} = 0$$

at least for one value of x_i , y_i , and such that

$$c_1^{(i)} A_{j1} + c_2^{(i)} A_{j2} + c_3^{(i)} A_{j3} \geq 0,$$

$$c_1^{(i)} A_{k1} + c_2^{(i)} A_{k2} + c_3^{(i)} A_{k3} \geq 0$$

for all values x_j , y_j and x_k , y_k respectively ($i, j, k = 1, 2, 3$; $i \neq j \neq k \neq i$). These conditions are sufficient for $n_i \geq n_i'$, $n_2 \geq n_2'$, $n_3 \geq n_3'$ (n_i' is a positive integer or zero; it is certainly zero when $q_i \leq 0$) if one replaces "at least for one value of x_i , y_i " by "takes both signs."

This paper is a part of an article to appear in the *Mathematische Annalen*.

23. Jordan and Dickson have investigated a system of linear groups in p^m variables, p being any prime, which contain invariant subgroups of order p^{2m+1} or p^{2m+2} . If regarded as a collineation group, any group of this system contains an invariant subgroup of order p^{2m} , and the quotient group is isomorphic with the special abelian linear group on $2m$ indices. From the existence of this system Dr. Mitchell proves the existence, for p an odd prime, of two systems of collineation groups in $(p^m \pm 1)/2$ variables, which are isomorphic with these quotient groups. All three systems have been investigated for $m = 1$ by Klein and Hurwitz and for $m = 2$ by Klein, Witting, and Burkhardt, particularly as regards their relation to elliptic and hyperelliptic functions. The last two do not seem to have been noticed before for $m > 2$.

24. Mr. Vandiver considers symmetric functions formed by a complete system of unit elements of a finite algebra, in particular the finite algebra formed by a complete system of residues of an ideal modulus in an algebraic field; and gives a proof of the theorem

$$\prod_U (x - U) \equiv \sum_{s=1}^k \rho_s (x^{\phi(P_s)} - 1)^{\phi(m)/\phi(P_s)} \pmod{m},$$

$$m = P_1^{\alpha_1} P_2^{\alpha_2} \cdots P_k^{\alpha_k}, \quad N(P_s) \neq 2, \quad (s = 1, 2, \dots, k),$$

the P 's being any prime ideals in an algebraic field U . $N(k)$ denotes the norm of k . Furthermore, ρ_s is an integer in the field such that $\rho_s \equiv 1 \pmod{P_s^{\alpha_s}}$ and $\rho_s \equiv 0 \pmod{m/P_s^{\alpha_s}}$. The product extends over a complete system of unit residues modulo m , and x is an indeterminate.

The discussion of functions of the residues of a rational modulus leads naturally to the arithmetical theories of Bernoulli's numbers and Fermat's quotient, and most of the known results relating to these are derived by a uniform process and generalizations are given.

25. In the first part of this paper Dr. Fischer gives a definition of the derivative of a function of a surface analogous to Volterra's definition of the derivative of a function of a curve. If the derivative exists and certain continuity conditions are satisfied, the first variation of the function is proved to be equal to the double integral of the product of this derivative and the variation of the variable z , the integration taking place over that part of the xy -plane over which the given surface is defined. It is also proved that the derivative must vanish identically over a surface which minimizes the value of the given function. In the last part of the paper only those surfaces are considered admissible which give fixed values to a given set of functions. The definition of the derivative of another function of one of these surfaces is modified appropriately and the theorems mentioned above are proved for this restricted set of surfaces also. When the functions considered are defined by means of double integrals the theory developed has an interesting application to the double integral problem of the calculus of variations.

26. In this paper Dr. Sullivan applies the theory of surfaces

as developed by Professor Wilczynski (*Transactions*, 1907 to 1912) to a study of the non-developable surfaces whose asymptotic curves belong to linear complexes.

It is found that: the ruled surfaces have straight line directrices, a result due to A. Peter but established by a method entirely different from that used here. The curved surfaces are the envelopes of either of two one-parameter families of ruled surfaces, having straight line directrices, which belong to complementary reguli of a directrix quadric. They are the integrating surfaces of a canonical system of non-involutory completely integrable partial differential equations of the second order characterized by the following equations for the fundamental invariants:

$$1. a' = 0, b = UV, \quad (\text{ruled surfaces}),$$

$$2. a' = b = \sqrt{U'V'}/(U + V), \quad (\text{curved surfaces}),$$

where U and V are arbitrary functions of the asymptotic parameters. A geometrical construction is found for surfaces having the property in question.

Special applications are made to certain organic curves and congruences of the surfaces, to the loci of the pinch points of the Cayley cubic scrolls osculating the ruled surfaces, and to the ruled surfaces whose asymptotic curves are twisted cubics.

29. Since geometries are classified according to their characteristic transformations, it would seem that a natural method of postulating a geometry would be in terms of point and transformation. This is done in Mr. Gaba's paper for general projective geometry. The basis is a set of elements called points and a set of transformations of points to points called collineations. Eight postulates upon points and collineations are given, from which are derived as theorems the axioms listed by Veblen and Young in their *Projective Geometry*.

30. Dr. Hurwitz states a slightly altered form of his set of two postulates for abelian groups (*Annals of Mathematics*, second series, volume 8 (1907), page 94), and with this as basis, constructs a set of five postulates for fields, thus reducing by two postulates the smallest number used hitherto. Consistency and independence are discussed for classes of countably infinite, continuously infinite, and specified finite numbers

of elements. As an incidental result, a simple type of finite commutative non-associative linear algebras is found, admitting unique division by non-zero elements, but not containing idempotent elements.

31. The Appell transformation, consisting of a projective transformation of the space coordinates together with a certain change in differential of the time, converts any field of force into another field. The trajectories of the two fields are projectively related, but the forces themselves are not so related. Professor Kasner finds the explicit geometric connection of the corresponding forces.

32. According to the general method of Professor Wilczynski, the projective geometry of a given configuration may be studied by means of a completely integrable system of differential equations, of which the integral equations defining the configuration form a fundamental system of solutions. In the present note, Mr. Green points out that in connection with a given completely integrable system of differential equations a number of different geometric configurations may be studied; in particular, he cites the projective geometry of certain systems of k -spreads in an r -spread immersed in an n -space. Thus, in ordinary space, the theories of triple systems of surfaces, of a single system of surfaces, and of a congruence of curves, are all founded on the consideration of the same completely integrable system of differential equations, proper transformations being made in each case.

33. Professor Young's paper is a modification and an extension of the paper presented by him to the Society, December 28, 1911, under the title "On algebras defined by groups of transformations" (see BULLETIN, volume 18 (February, 1912), page 227). In the present paper he considers a system (Σ, \mathcal{G}) , where Σ is a general class of elements (a, b, c, \dots) and \mathcal{G} is a set of transformations on Σ to Σ , with the property that corresponding to every element a of Σ there exists uniquely a transformation G_a of \mathcal{G} . He defines an operation \circ by the relation $G_a(b) = a \circ b$. \mathcal{G}_a is then represented by the relation $x' = a \circ x$, where x is a variable ranging over Σ ; and \mathcal{G} is the set of transformations $x' = a \circ x$, where the parameter a ranges over Σ . The relations $x' = x \circ a$ constitute the adjoint set $\bar{\mathcal{G}}$ of

transformations. If \mathfrak{G} is a group, $\overline{\mathfrak{G}}$ is also a group, called the adjoint group. The operation \circ thus defined between ordered pairs of elements of Σ is very general. It may or may not be one-valued, or associative, or commutative; the inverse operations may or may not exist and, if existent, may or may not be one-valued; there may or may not exist identity elements (i. e., elements i_r or i_l such that $a \circ i_r = a$ or $i_l \circ a = a$ for every a); etc. These and similar properties are implied by certain properties of the system (Σ, \mathfrak{G}) , which may be stated readily in the terminology of the theory of transformations. The author lists fourteen such properties and analyses their logical interdependence.

An algebra is obtained by the combination of two or more such elementary systems (Σ, \mathfrak{G}) which are connected; i. e., by a system $(\Sigma_1, \mathfrak{G}_1; \Sigma_2, \mathfrak{G}_2; \dots)$ in which $\Sigma_1, \Sigma_2; \dots$ have common elements. Properties connecting two or more elementary systems of an algebra lead to relations connecting the corresponding operations. Thus necessary and sufficient conditions are obtained in order that one operation be distributive with respect to another, either in the ordinary or in a generalized sense (cf. abstract of earlier paper cited above).

The theory thus developed may readily be applied in various directions. As a very elementary application von Staudt's algebra of points on a line is obtained. By considering operations defined by systems (Σ, \mathfrak{G}) in which Σ are the points of a projective plane and \mathfrak{G} is a group of collineations, the author shows that the adjoint group may be either a group of collineations, or a group of Cremona transformations of degree n , or a group of transcendental transformations. He applies the results to the determination of all point algebras in a plane having two operations $+$ and \cdot satisfying the associative, commutative, and distributive laws of ordinary algebra, in which the transformations $x' = a + x$ and $x' = a \cdot x$ are collineations. He finds that in the complex plane there are only two essentially distinct algebras of this kind; whereas in the real plane there are three such algebras, one of which is, of course, isomorphic with the algebra of ordinary complex numbers.

34. In a paper presented to the Society, December 28, 1911, entitled "a generalization to 3-space and to n -space of the inversion geometry in a plane" (cf. BULLETIN, volume 18

(1912), page 229), Professor Young called attention to a set of groups \mathcal{G}_n of Cremona transformations in n -space, which for $n = 2$ reduced to the group of direct circular transforms in a plane. In the present paper Professor Young and Dr. Morgan, after indicating a further generalization of these groups, present a more systematic investigation of the geometry defined by \mathcal{G}_3 . This group is defined geometrically as follows: Given a triangle $U_1U_2U_3$ (which for convenience of reference is chosen at infinity) and a projectivity in each of the pencils of planes on the sides of this triangle, a point P (not at ∞) determines uniquely a plane in each of these pencils; if the projectivity transforms these planes into three planes meeting in a point P' , P and P' are equivalent under a transformation of \mathcal{G}_3 . All the transformations of \mathcal{G}_3 are obtained by varying in all possible ways the projectivities in the pencils of planes referred to.

The "space" of the geometry consists of all "ordinary" points, that is points in the ordinary sense not at ∞ ; and of all "ideal" points defined as follows: a point U_i ($i = 1, 2, 3$) with any line through U_i , not at ∞ , is an ideal point of the first kind; a side of the triangle $U_1U_2U_3$ and any plane through this side is an ideal point of the second kind; the plane at infinity is the ideal point of the third kind. In this space the transformations of \mathcal{G}_3 are one-to-one without exception. The transformations are cubic except when an ideal point of the second kind is invariant, in which case they are quadratic; and when ∞ is invariant, in which case they are linear. The simple one-dimensional elements of the geometry, the so-called characteristic curves, are in general twisted cubics through U_1 , U_2 , and U_3 . For further details see abstract of earlier paper cited above. The simple two-dimensional elements of the geometry, the so-called characteristic surfaces, are: (1) any cubic surface with conical points at U_1 , U_2 , and U_3 and one free conical point (the "vertex" of the surface). If the vertex of the surface is an ideal point of the first kind at U_i the surface has a binode at U_i ; (2) any quadric cone through U_1 , U_2 , and U_3 with vertex (the "vertex" of the surface) at an ordinary point or on a side of the triangle $U_1U_2U_3$ (in the latter case the "vertex" is an ideal point of the second kind); (3) any plane not on U_1 , or U_2 , or U_3 (in this case the "vertex" is the point ∞). Among the theorems derived may be mentioned: any two characteristic surfaces are equivalent under

③. The vertex and any other three points (in general position) of a characteristic surface uniquely determine the surface. A characteristic curve through the vertex and two other points of the surface lies completely on the surface.

F. N. COLE,
Secretary.

THE TWENTY-THIRD REGULAR MEETING OF THE SAN FRANCISCO SECTION.

THE twenty-third regular meeting of the San Francisco Section of the Society was held at the University of California, on Saturday, April 12, 1913. The following members of the Society were present:

Mr. B. A. Bernstein, Professor H. F. Blichfeldt, Dr. Thomas Buck, Professor G. C. Edwards, Mr. W. F. Ewing, Professor M. W. Haskell, Professor L. M. Hoskins, Dr. Frank Irwin, Professor D. N. Lehmer, Professor W. A. Manning, Professor H. C. Moreno, Dr. L. I. Neikirk, Professor E. W. Ponzer, Professor T. M. Putnam.

Professor G. C. Edwards presided at both morning and afternoon sessions. It was decided to hold the next regular meeting of the Section at Stanford University, on October 25, 1913.

The following program was presented:

(1) Mr. B. A. BERNSTEIN: "A set of postulates for the algebra of positive rational numbers with zero."

(2) Professor H. F. BLICHFELDT: "On the arithmetic value of quadratic forms."

(3) Dr. L. I. NEIKIRK: "The analytical geometry of functional space."

(4) Professor T. M. PUTNAM: "Concerning the residues of certain sums of powers of integers to a prime modulus."

(5) Dr. H. W. STAGER: "A geometrical transformation, with some applications to certain systems of spheres."

Abstracts of the papers follow below.

1. In volume 3 of the *Transactions*, Professor Huntington gives a set of postulates for the positive rational numbers. By modifying Professor Huntington's set, Mr. Bernstein obtains a complete set of postulates for the algebra of positive rational numbers with zero.