may also be done, but this may involve greater difficulties, by imposing appropriate conditions on the coefficients of the system.

Thus, *the projective geometry of an analytic r-spread in a linear space of n dimensions is equivalent to the theory of the invariants and covariants of a completely integrable system of linear partial differential equations with r independent variables, whose general solution depends on* $n + 1$ *arbitrary constants.*

If we recall our preliminary discussion regarding the arbitrariness of the space element, and the great generality which is therefore involved in the notion "r-spread in n dimensions" even as applied to ordinary space, we shall appreciate the sweeping character of this generalization which unifies such a vast domain. To the mathematician who knows that metric properties may, in a certain sense, be regarded as projective properties, it will be evident what must be added in order that this unifying principle may embrace metric geometry as well.

THE UNIVERSITY OF CHICAGO,
December, 1912.

ON CERTAIN NON-LINEAR INTEGRAL EQUATIONS

BY MR. H. GALAJIKIAN.

(Read before the American Mathematical Society, December 31, 1912.)

NON-LINEAR integral equations of the Volterra type have been considered by Lalesco,* Cotton,† and Picone.‡ The two theorems of the present paper give results which are of more general character. Theorems apparently still more general have been stated very recently by Evans.§ The method used is that of successive approximations. The plan of treatment applies to integral equations of the type

* *Journal de Mathématiques*, series 6, vol. 4 (1908), p. 165; Introduction à la Théorie des Equations Intégrales, p. 127.

† *Bulletin de la Société math. de France*, vol. 38 (1910), p. 144.

‡ *Rendiconti del Circolo Matem. di Palermo*, vol. 30 (1910), p. 351.

§ Proceedings of the International Congress of Mathematicians, Cambridge, December, 1912. The present paper was completed without knowledge of Professor Evans' work, and forms one section of a Cornell University master's thesis, which was officially approved in May, 1912.

$$y(x) = g\left\{x, \int_{x_0}^{x} f_1[x, t, y(t)]dt, \cdots, \int_{x_0}^{x} f_m[x, t, y(t)]dt\right\},$$

or to a system of n such equations with n unknown functions; as a very special case we have the usual existence theorems for systems of ordinary differential equations. For brevity we consider, as a typical case, the integral equation

$$(1) \qquad y(x) = g\left\{x, \int_{x_0}^{x} f_1[x, t, y(t)]dt, \int_{x_0}^{x} f_2[x, t, y(t)]dt\right\}.$$

THEOREM I. *Let $g\{x, u_1, u_2\}$ satisfy a Lipschitz condition*

$$|g\{x + \xi, u_1 + \mu_1, u_2 + \mu_2\} - g\{x, u_1, u_2\}| \leqq A\{|\xi| + |\mu_1| + |\mu_2|\},$$

in the region $|x - x_0| \leqq \alpha$, $|u_1| \leqq \beta$, $|u_2| \leqq \beta$. Let $f_1(x, t, y)$ and $f_2(x, t, y)$ be continuous, and satisfy a Lipschitz condition for the argument y

$$|f_i(x, t, y + \eta) - f_i(x, t, y)| \leqq B|\eta| \qquad (i = 1, 2),$$

in the region $|x - x_0| \leqq \alpha$, $|t - x_0| \leqq \alpha$, $|y - g\{x_0, 0, 0\}| \leqq \gamma$; furthermore let

$$|f_i(x, t, y)| \leqq M \qquad (i = 1, 2).$$

Then if ρ satisfies the conditions

$$\rho \leqq \alpha, \quad \rho \leqq \frac{\beta}{M}, \quad \rho \leqq \frac{\gamma}{A(1 + 2M)},$$

there exists, in the interval $|x - x_0| \leqq \rho$, one and only one continuous solution $y(x)$ of the integral equation (1).

Define the functions

$$y_0(x) = g\{x_0, 0, 0\}$$

$$(2) \quad y_n(x) = g\left\{x, \int_{x_0}^{x} f_1[x, t, y_{n-1}(t)]dt, \int_{x_0}^{x} f_2[x, t, y_{n-1}(t)]dt\right\} \qquad (n = 1, 2, \cdots).$$

At each stage of the approximation, we see that the function

$y_n(x)$ satisfies the conditions

$$| y_n(x) - g\{x_0, 0, 0\}| \leqq \gamma,$$

$$\left| \int_{x_0}^{x} f_i[x, t, y_n(t)]dt \right| \leqq \beta \qquad (i = 1, 2),$$

when $|x - x_0| \leqq \rho$; hence the next following approximation will have a meaning. Evidently each $y_n(x)$ is continuous, $|x - x_0| \leqq \rho$.

We shall prove that the sequence $y_n(x)$ approaches a limit uniformly. To this end write

$$y_n(x) - y_{n-1}(x) = Y_n(x) \qquad (n = 1, 2, \cdots),$$

so that

$$y_n(x) = y_0(x) + Y_1(x) + Y_2(x) + \cdots + Y_n(x) \qquad (n = 1, 2, \cdots).$$

The usual methods suffice to show that

$$|Y_n(x)| \leqq \frac{2^{n-1}A^nB^{n-1}(1 + 2M)|x - x_0|^n}{n!},$$

hence that

$$|Y_n(x)| \leqq \frac{2^{n-1}A^nB^{n-1}(1 + 2M)\rho^n}{n!} = \frac{1 + 2M}{2B}\frac{(2AB\rho)^n}{n!}$$

when $|x - x_0| \leqq \rho$. Since the series of constants

$$\sum_{n=1}^{\infty} \frac{(2AB\rho)^n}{n!}$$

converges, we see that the series $y_0(x) + Y_1(x) + \cdots + Y_n(x) + \cdots$ converges uniformly, $|x - x_0| \leqq \rho$; if we represent its value by $y(x)$ we have

$$\lim_{n=\infty} y_n(x) = y(x)$$

uniformly, $|x - x_0| \leqq \rho$. Thus $y(x)$ is a continuous function which, as we see by reference to (2), satisfies the integral equation (1).

That there is only one such continuous solution may be seen as follows. Suppose there were two, $y(x)$ and $z(x)$, and put $y(x) - z(x) = w(x)$. Then if we write equation (1) for $y(x)$ and $z(x)$, and subtract, we see readily from the given conditions on the functions g, f_1, f_2, that

$$|w(x)| \leqq 2AB \left| \int_{x_0}^{x} |w(t)| dt \right|.$$

If now we write $|w(x)| \leqq W$, we find by successive applications of the preceding formula

$$|w(x)| \leqq 2ABW|x - x_0|,$$

$$|w(x)| \leqq \frac{4A^2B^2W|x - x_0|^2}{2!},$$

and in general,

$$|w(x)| \leqq \frac{W(2AB|x - x_0|)^n}{n!}.$$

Since the expression on the right has the limit zero as n becomes infinite, we see that $w(x) = 0$; thus $y(x) = z(x)$.

It is interesting to note that if we replace the condition

$$|y(x) - g\{x_0, 0, 0\}| \leqq \gamma$$

on the arguments of f_1, f_2, by the condition

$$|y(x) - g\{x, 0, 0\}| \leqq \gamma,$$

we may broaden the results of the theorem in two ways: first, in that we impose on the function $g\{x, u_1, u_2\}$ a Lipschitz condition involving only the last two arguments, instead of all three; secondly, in that we give the solution in a less restricted interval. The facts are stated as follows:

THEOREM II. *Let $g\{x, u_1, u_2\}$ be continuous, and satisfy a Lipschitz condition for the arguments u_1, u_2*

$$|g\{x, u_1 + \mu_1, u_2 + \mu_2\} - g\{x, u_1, u_2\}| \leqq A\{|\mu_1| + |\mu_2|\}$$

in the region $|x - x_0| \leqq \alpha$, $|u_1| \leqq \beta$, $|u_2| \leqq \beta$. Let $f_1(x, t, y)$ and $f_2(x, t, y)$ be continuous, and satisfy a Lipschitz condition for the argument y

$$|f_i(x, t, y + \eta) - f_i(x, t, y)| \leqq B|\eta| \qquad (i = 1, 2)$$

in the region $|x - x_0| \leqq \alpha$, $|t - x_0| \leqq \alpha$, $|y - g\{x, 0, 0\}| \leqq \gamma$;

furthermore let

$$|f_i(x, t, y)| \leqq M \qquad (i = 1, 2).$$

Then if ρ *satisfies the conditions*

$$\rho \leqq \alpha, \quad \rho \leqq \frac{\beta}{M}, \quad \rho \leqq \frac{\gamma}{2AM},$$

there exists, in the interval $|x - x_0| \leqq \rho$, *one and only one continuous solution* $y(x)$ *of the integral equation* (1).

The proof is entirely similar to that of Theorem I.

CORNELL UNIVERSITY,
January 10, 1913.

A THEOREM ON ASYMPTOTIC SERIES.

BY MR. VINCENT C. POOR.

THEOREM: *If* $f(z)$ *is not holomorphic at* $z = 0$, *but is formally developable into a Maclaurin series, and if* w *is asymptotic to* $a_1/z + a_2/z^2 + \cdots$ (*written:* $w \backsim a_1/z + a_2/z^2 + \cdots$), *then* $f(w)$ *has an asymptotic representation.**

To prove this theorem take $f(z)$ in the form

$$(1)\qquad f(z) = f(0) + f'(0)z + \frac{f''(0)}{2!}z^2 + \cdots + \frac{f^{(n)}(0)z^n}{n!} + \int_0^z f^{(n+1)}(t)\cdot\frac{(z-t)^n}{n!}\,dt.$$

Since

$$w \backsim \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots$$

w may be written

$$(2)\qquad w = \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots + \frac{a_n + \epsilon_{1nz}}{z^n},$$

where, according to the Poincaré definition† for an asymptotic

---

* This theorem is the " résultat préalablement obtenu " referred to in Professor Ford's paper in the *Bulletin* of the French Society for 1911. See *Bulletin Société math. de France*, vol. 40 (1912), fascicule 1 under " Erratum du Tome XXXIX."

† Poincaré, *Acta Mathematica*, vol. 8 (1886), p. 296.