SOME GENERAL ASPECTS OF MODERN GEOMETRY.*

BY PROFESSOR E. J. WILCZYNSKI.

IT is a great honor and an exceptional privilege to be asked to address such a distinguished audience as is assembled here upon this occasion. And so my first duty is the simple and elementary one of expressing to the officers of the societies, meeting in joint session, my gratitude for having selected me for such a task. But the task itself is not a simple one. Unwelcome as it may be, the fact remains that the workers in the fields of mathematics, physics, and astronomy, intimate associates in former times, have become comparative strangers. So widely have their various dialects diverged from the common mother tongue, that they find it possible to follow each other's speech only when great care is taken to articulate distinctly, and even then only at the expense of most intense and rigid attention. But while we may find it difficult to understand each other, after all, these sister sciences have much in common. The love and respect which they bear each other are still alive. They appreciate fully how great are the services which they can render each other, and how fruitful are those domains of thought in which these various subjects are made to intermingle. It is well that we should specialize, for only by intense application of intellectual forces to specific problems can real progress in science be made. But, unless we preserve a broad interest in a larger field, we run into the danger of losing a proper sense of balance and perspective. It is not true, even in science, that all things are of equal value, and it is better for science that we should study important problems rather than unimportant ones. But which problems *are* important, and which are not? Here is a question which is worth some thought. We know that it cannot be answered from the utilitarian point of view, at least not in an adequate and permanent fashion. We also know that any attempt to impose upon each other our *individual* criterion of value can only

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result in harm. It is well then that we should meet and discuss our problems, that we should attempt to formulate into general principles the results of our daily meditations, in the hope that these principles, whose value has been tested in our own individual experience, may prove to be helpful in some other related field. This is the interpretation which I have placed upon my task, an attempt to show how one great unifying principle pervades the whole realm of geometry.

The distinction between analysis and geometry, while a convenient one, is really superficial and in some respects injurious. For, if there is any one thing which the invention of analytic geometry has taught us, it is this: that every problem of analysis is capable of a geometrical interpretation, and that every problem of geometry may be formulated analytically. It is my individual conviction that no mathematical investigation is truly complete unless it puts into evidence the existing relations both from an analytic and a geometric point of view. It is true that geometric intuition has occasionally led mathematicians into error, but the first intuitions of analysts have also frequently been found wanting. *All* of our intuitions must be subjected to rigorous criticism. Geometry obeys the same laws of logic as analysis, and the disrepute into which it has fallen, in some quarters, is due to the fact that naive geometric intuitions have been compared, to their disadvantage, with refined analytical theories rather than with the naive analytic intuitions to which they really correspond. But, aside from the question of rigor, it is very important that our mathematical theorems should present themselves to us, *not* merely as the final consequences of long and complicated arguments. They are not truly our own, we have not fully seized their significance, until from some point of view they appear to be obviously and intuitively true. In very many cases, geometry furnishes the best method for thus intuitively grasping the full import of a mathematical situation. And this is true, not merely in the case of rough and simple analogies, but also in those very cases in which an untrained and naive intuition had caused the earlier students to go astray. Thus, for instance, the theory of uniform convergence, as presented by Osgood in geometric form, assumes a convincing force which no mere analytic treatment could give it, although the logic is precisely the same whether the argument be presented analytically or geometrically.

There is then, to my mind, no fundamental distinction between geometry and analysis. If, nevertheless, I have used the word geometry in the title of this address, I have done so because, according to the traditional classification, the questions which I shall discuss are generally regarded as questions of geometry.

The peculiar efficacy of geometric notions for illustrating an abstract argument has given rise to a striking paradox. The elements of analytic geometry have taught us to associate with a point in a plane a pair of numbers, and with a point of space a system of three numbers, the coördinates of the point. The desire for complete parallelism between analysis and geometry has led to the notion of a point in a space of n dimensions as the geometric image of a system of n numbers $(x_1, \cdots, x_n)$. Although we surely cannot be said to have any primitive intuitive notions as to the properties of a space of n dimensions, we nevertheless speak of curves, surfaces, etc., in such a space, the analogies indicated by this geometric manner of speech being extremely valuable and suggestive for the purposes of analysis. Thus, and this is the paradox to which I am referring, we make use of the abstract idea of an n-space, of which we have no direct geometric intuition, to render intelligible so concrete a thing as a system of n numbers.

The idea of a space of n dimensions (or an n-space) has now become an essential part of our mathematical patrimony. The notion developed gradually, and traces of it are to be found in the writings of several of the mathematicians of the latter part of the eighteenth and the early part of the nineteenth century, especially in those of Gauss and Cauchy. The complete notion of an n-space, however, with all of its most essential implications, must be ascribed to Grassmann, whose first "Ausdehnungslehre" of 1844 is largely devoted to this subject.

It seems, at first thought, as though the dimensionality of a space ought to be regarded as its most important characteristic, and that it would be vain to attempt to look at the same space in two different fashions, so as to attribute to it two different dimensionalities. It seems, also, as though nothing could be more hopeless than to attempt to associate a genuine geometric intuition with such an abstract notion as that of an n-space. And indeed, if all of our thinking were abstract, I doubt whether the possibility of doing either of these things would ever have occurred to anybody. From another point of view, however, which gradually presented itself in the course of development of mathematical thought, the affirmative answers to both of these questions become so evident as to appear almost trivial. Two geometers of the early part of the nineteenth century, Poncelet and Gergonne, had discovered what is now known as the principle of duality, according to which every theorem of projective geometry may be made to yield a second one by the simple process of interchanging the words point and plane, and leaving the word line unchanged. It thus became apparent that, for the purposes of projective geometry, the point and the plane were coördinate notions. To the already existing ideas of curves and surfaces thought of as point loci, were added the strictly correlative notions of one and two-dimensional aggregates of planes and their respective envelopes. Thus, as a consequence of the principle of duality, for the first time an element different from a point, namely a plane, was thought of as the generating unit of geometric forms. But our ordinary space is three-dimensional from the point of view of its planes as well as of its points, so that the dimensionality of space was still left unchanged. Moreover, this single instance of a change of the space element was too isolated and special a thing to inspire any easy or far-going generalization. It was a great step in advance, therefore, when Plücker in 1846 proposed to regard the straight line as the generating element of space, and introduced the notion of line coördinates. For here, for the first time, do we find space presenting itself as a four-dimensional aggregate, thus destroying the idea of dimensionality as an inherent geometric characteristic of space. And here too, do we find the fountain head for all of those generalizations of modern geometry, in which not merely the point, plane, and line, but countless other geometric forms appear as generating elements. There is nothing easier, nowadays, than to represent concretely in the plane, a geometry of any number of dimensions.

For the purpose of characterizing some particular branch of geometric research, the choice of the space element is particularly important. The same analytic theorems may, by a change of space element, receive many widely differing geometrical interpretations. The principle of duality, to which I have already alluded, is probably the best known illustration of this fact. Another system of abstractly equivalent geometric theories is given by Plücker's line geometry, Lie's sphere geometry, and the geometry of a quadric four-spread in a space of five dimensions. Similarly it is, abstractly speaking, the same thing whether we are discussing a linear space of five dimensions, the aggregate of conics in a plane, or the totality of linear complexes in ordinary space. But the geometric content of our theorems is very different in these various cases. It is not necessarily desirable, even in any particular investigation, to consider always the same geometric form as space element. In fact, one of the most fruitful results of the discussions of Plücker and his successors is the freedom which they have given to the present day geometer to change his space element whenever the change may seem desirable.

Let us suppose that we have selected some geometric form as generating element for a particular geometric theory which we wish to develop. If this element requires k numbers for its complete determination we may, in accordance with our previous remarks, speak of it as a "*point*" in a space of k dimensions, this *space* being the aggregate of all such elements. This *space* may have the property that every point of the line which joins any two of its points itself belongs to this space.* If this is so we shall call it a linear space. Such is, for instance, the two-dimensional aggregate of all of the points of a plane, the three-dimensional aggregate of all points of ordinary space, the five-dimensional aggregate of all conics in a plane. As an illustration of a non-linear two-dimensional space we may take the aggregate of all of the points of a curved surface. Such a curved surface may, however, be regarded as immersed in a linear space of three dimensions, and indeed this is our customary way of looking at it. In the same way, if the space of k dimensions determined by our space element is not a linear space, we shall think of it as immersed in a linear space of $n > k$ dimensions, choosing the number n as small as may be compatible with the nature of our original non-linear k-space. Now a point of this linear n-space is determined by n coördinates $x_1, \cdots, x_n$. But since the aggregate of all of the geometric forms which we are using as space elements has only k dimensions we shall have to think of $x_1, \cdots, x_n$ as

* Of course the application of this criterion presupposes a definition of the word line. But we cannot discuss such details in this address.

satisfying $n - k$ independent equations none of which are of the first degree, since otherwise our k-dimensional aggregate of space elements would belong to a linear space of less than n dimensions. In most applications these $n - k$ equations are algebraic.

In the language of hypergeometry we are then dealing with a point upon an algebraic k-spread immersed in a space of n dimensions. This k-spread is characterized by the $n - k$ independent algebraic equations

$$(1) \qquad f_i(x_1, x_2, \cdots, x_n) = 0 \quad (i = 1, 2, \cdots, n - k),$$

none of which is of the first degree.*

Let us consider, first, the case that no such equations are present, so that all of the points of our n-space are available as generating elements for the geometric forms which we wish to study. Any system of values $(x_1, \cdots, x_n)$ gives us a "point" of such a space; several such systems give us several "points." We are primarily interested in the case where we have an infinite number of such points. If we assume that the points of such an infinite set form a continuous analytic aggregate, we shall have expressions of the form

$$(2) \qquad x_1 = \varphi_1(u_1, \cdots, u_r), \quad \cdots, \quad x_n = \varphi_n(u_1, \cdots, u_r)$$

for their coördinates, where r may be any integer between 1 and n, and where $\varphi_1, \cdots, \varphi_n$ are analytic functions of their arguments. If $r = 1$ we have a one-dimensional spread, or curve, composed of a single infinity of points of our n-space S_n. If $n = 2$ we have a two-dimensional spread, or surface, immersed in S_n. In every case we find an analytic r-spread immersed in our space S_n of n dimensions, which for $r = n$ coincides with S_n itself, or at least with an n-dimensional portion of S_n.

If we use three coördinates for a point in ordinary space, some complications arise, caused by the exceptional rôle played by the "points at infinity." To avoid this difficulty it has long been customary to introduce the so-called homogeneous coördinates. For precisely the same reason it will be advantageous to introduce homogeneous coördinates for the points of our n-space. Let us write

* If the k-spread is to be irreducible, it may require more than $n - k$ equations to characterize it completely.

$$x_1 = \frac{y_1}{y_{n+1}}, \quad x_2 = \frac{y_2}{y_{n+1}}, \quad \cdots, \quad x_n = \frac{y_n}{y_{n+1}},$$

i. e., let us introduce a system of $n + 1$ numbers $y_1, y_2, \cdots, y_{n+1}$ whose ratios are equal to $x_1, x_2, \cdots, x_n$ respectively. These $n + 1$ numbers, only whose ratios are of interest for us, are called the homogeneous coördinates of the point. The homogeneous coördinates of any point of our r-spread will then be given by $n + 1$ equations of the form

$$(3) \qquad \begin{aligned} &y_1 = \psi_1(u_1, \cdots, u_r), \quad y_2 = \psi_2(u_1, \cdots, u_r), \quad \cdots, \\ &\qquad\qquad\qquad\qquad y_{n+1} = \psi_{n+1}(u_1, \cdots, u_r), \end{aligned}$$

the geometrical content of which equations would not be altered if we were to multiply all of their right members by any common factor $\lambda(u_1, \cdots, u_r)$, since such a multiplication obviously has no influence upon the values of the ratios $y_1 : y_2 : \cdots : y_{n+1}$. We may assume, of course, that the functions $y_1, \cdots, y_{n+1}$ are linearly independent. For, if they were not, we could reduce our problem to a similar one in a linear space of fewer than n dimensions.

We wish to show that we can always find a system of linear homogeneous differential equations of which $y_1, y_2, \cdots, y_{n+1}$ are the fundamental solutions, in the sense that the most general solution of the system will have the form

$$(4) \qquad y = c_1 y_1 + c_2 y_2 + \cdots + c_{n+1} y_{n+1},$$

where $c_1, c_2, \cdots, c_{n+1}$ are arbitrary constants.

The truth of this statement is obvious for $r = 1$. We are then dealing with a *curve* of S_n, and $y_1, y_2, \cdots, y_{n+1}$ may be regarded as the fundamental solutions of an ordinary linear homogeneous differential equation of the $(n + 1)$th order

$$(5) \qquad \frac{d^{n+1}y}{du^{n+1}} + p_n(u)\frac{d^n y}{du^n} + \cdots + p_1(u)\frac{dy}{du} + p_0(u)y = 0,$$

whose general solution will then be given by (4).

If $r > 1$ we shall have to consider partial differential equations. A function of r independent variables has r partial derivatives of the first order, $\frac{1}{2}r(r + 1)$ partial derivatives of the second order, etc., $[r(r + 1) \cdots (r + k - 1)]/k!$ partial derivatives of order k. We may think of the function itself as its zeroth derivative. Thus there will be altogether

$$(6) \qquad p_k = 1 + \sum_{i=1}^{k} \frac{r(r+1)\cdots(r+i-1)}{i!}$$

partial derivatives of a function y of $u_1, \cdots, u_r$, whose order does not exceed k. This number grows very rapidly with k, and we may obviously choose k so large that p_k shall become greater than $n + 1$, n being the number of dimensions of the space under consideration.

If our r-spread does not degenerate into an $r - 1$-spread, $y_1, \cdots, y_{n+1}$ cannot satisfy one and the same linear homogeneous partial differential equation of the first order

$$Ay + B_1 \frac{\partial y}{\partial u_1} + \cdots + B_r \frac{\partial y}{\partial u_r} = 0.$$

For, if they did, the ratios of $y_1, y_2, \cdots, y_{n+1}$ would be functions of at most $r - 1$ combinations of $u_1, \cdots, u_r$, i. e., we should be at most dealing with an $(r - 1)$-spread.

All of the functions $y_1, \cdots, y_{n+1}$ may, however, satisfy one or several such partial differential equations of the second order. If they satisfy as many as $\frac{1}{2}r(r+1)$ independent equations of this kind, *all* of the second order derivatives can be expressed in the form

$$\frac{\partial^2 y}{\partial u_i \partial u_k} = A_{ik} y + B_{i,k}^{(1)} \frac{\partial y}{\partial u_1} + \cdots + B_{i,k}^{(r)} \frac{\partial y}{\partial u_r} \qquad (i, k = 1, 2, \cdots, r),$$

and therefore also all of the derivatives of higher order. The most general analytic solution of such a system is clearly a linear homogeneous combination with constant coefficients of $r + 1$ independent ones, so that our r-spread must be contained in a linear r-space. Since we have assumed that the n-space, which we have under consideration, is the linear space of lowest dimensionality which contains our r-spread, this case can present itself only if $r = n$.

In general, our $n + 1$ functions $y_1, \cdots, y_{n+1}$ will satisfy fewer than $\frac{1}{2}r(r+1)$ linear homogeneous partial differential equations of the second order, perhaps none at all. If the number of such equations is $\frac{1}{2}r(r+1) - s$, we may regard

y, $\partial y/\partial u_1, \cdots, \partial y/\partial u_r$, and s of the second order derivatives as linearly independent while the remaining second order derivatives are expressible linearly and homogeneously in terms of these $1 + r + s$ quantities with coefficients which may be functions of $u_1, \cdots, u_r$. If $1 + r + s$ is less than $n + 1$, we examine the derivatives of the third order. Suppose that all of these are expressible linearly and homogeneously in terms of the above $1 + r + s$ quantities and of t independent third order derivatives. Let us continue this process. We shall finally have all of the partial derivatives of a certain, say the kth, order expressed linearly and homogeneously in terms of $1 + r + s + \cdots + w$ of them, where k is so large that for the first time

$$1 + r + s + \cdots + w \geqq n + 1. \tag{7}$$

For, if this were not so, we could express all of the partial derivatives, of all orders, linearly and homogeneously in terms of less than $n + 1$ independent ones and our r-spread would be contained in a linear space of fewer than n dimensions, contrary to our hypothesis.

But the sum $1 + r + s + \cdots + w$ cannot exceed $n + 1$. For, if it did, our r-spread could not be contained in any linear n-space. Therefore we have

$$1 + r + s + \cdots + w = n + 1. \tag{8}$$

We have found a system of linear homogeneous differential equations, consisting of $\frac{1}{2}r(r + 1) - s$ equations of the second order, $\frac{1}{3}r(r + 1)(r + 2) - t$ equations of the third order, etc., $[r(r + 1) \cdots (r + k - 1)]/k! - w$ equations of the kth order. In most cases, these equations and those obtained from them by differentiation will enable us to express all of the derivatives of order $k + 1$ linearly and homogeneously in terms of the $n + 1$ independent ones of lower order, and the same thing will then be true of all derivatives of order $k + 2$, $k + 3$, etc. If, however, some of the derivatives of order $k + 1$ cannot be determined in this way, we can always add to our system a sufficient number of equations of order $k + 1$ of which $y_1, \cdots, y_{n+1}$ will also be solutions, to insure that all derivatives of order $k + 1$ will appear as linear homogeneous functions of the $n + 1$ fundamental derivatives of lower order.

The coefficients of this system will be analytic functions of

$u_1, \cdots, u_r$ and its most general solution will be of the form

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_{n+1} y_{n+1}, \tag{9}$$

where $c_1, \cdots c_{n+1}$ are arbitrary constants.

Such a system of partial differential equations is called a completely integrable system.

Obviously, if any completely integrable system is given, its solutions may be interpreted as the coördinates of the points of an r-spread in n dimensions. We see, therefore, that to every analytic r-spread contained in a linear space of n dimensions, there corresponds a completely integrable system of linear homogeneous partial differential equations whose general solution contains $n + 1$ arbitrary constants, and conversely.

But we can give a more precise significance to our result. The r-spread with which we started is not the only one which satisfies our completely integrable system of equations. The r-spread, whose equations are

$$\begin{aligned} \bar{y}_1 &= c_{11} y_1 + c_{12} y_2 + \cdots + c_{1,\, n+1}\, y_{n+1}, \\ &\cdots\cdots\cdots\cdots \\ \bar{y}_{n+1} &= c_{n+1,\, 1} y_1 + c_{n+1,\, 2} y_2 + \cdots + c_{n+1,\, n+1}\, y_{n+1}, \end{aligned} \tag{10}$$

where the quantities c_{ik} are arbitrary constants with a non-vanishing determinant, will also satisfy the same system of partial differential equations. Moreover, since we know that (9) is the expression for the most general solution of our system, no r-spreads, other than those expressible by (10), will satisfy the same system of partial differential equations. Now the equations (10) are precisely the equations of the most general projective transformation of our n-space, projective transformations being those which convert every linear k-spread of the space again into a linear k-spread. Thus, by means of our system of partial differential equations alone, we shall not be able to distinguish between the original r-spread and any one of its projective transformations. The properties which are common to all of these projectively equivalent r-spreads are called projective properties. Consequently our completely integrable system of equations, taken by itself, is concerned only with the projective properties of the r-spread.

However, the analytical representation of our r-spread given by equations (3), and consequently the resulting system of partial differential equations, contains some elements which cannot fairly be said to belong to the r-spread itself, and which may be changed without giving rise to a corresponding change in the r-spread. In fact, as we have already noticed, we may multiply $y_1, \cdots, y_{n+1}$ by a common factor $\lambda(u_1, \cdots, u_r)$ without changing the r-spread, since $y_1, \cdots, y_{n+1}$ are homogeneous coördinates. Furthermore an arbitrary transformation of the form

$$v_k = \varphi_k(u_1, u_2, \cdots, u_r) \qquad (k = 1, 2, \cdots, r)$$

merely changes the parameters to which the r-spread is referred, without affecting the r-spread itself. We are thus led to transform our system of partial differential equations by the transformations

$$(T) \qquad \begin{aligned} \eta &= \lambda(u_1, \cdots, u_r)y, \\ v_k &= \varphi_k(u_1, \cdots, u_r) \quad (k = 1, 2, \cdots, r), \end{aligned}$$

where the functions $\lambda, \varphi_1, \cdots, \varphi_r$ are arbitrary functions of their arguments. All of the systems obtained in this way from a given one correspond to the same class of projectively equivalent r-spreads. Those combinations of the coefficients and of the variables of our system of partial differential equations which are left unchanged when we make any transformation of the form (T) are called its *invariants* and *covariants*. Their values give the true and adequate expression of the projective properties of the r-spread, in a form independent of the accidental elements of any particular analytic representation.

We have discussed, so far, the case that no equations of the form (1), initially limiting us to the points of an algebraic k-spread of our n-space, are present. But it makes little difference for our theory if such equations do appear. Since our r-spread must then be contained in the k-spread whose equations are given by (1), the functions $y_1, \cdots, y_{n+1}$ will of their own accord satisfy these equations. If, however, instead of starting out with the explicit equations of a given r-spread, we were to begin our theory with a given completely integrable system of partial differential equations, we should have to impose upon its solutions the condition of satisfying the conditions (1). This may be done, very simply, by adding these equations (1) as subsidiary conditions to our system. It may also be done, but this may involve greater difficulties, by imposing appropriate conditions on the coefficients of the system.

Thus, *the projective geometry of an analytic r-spread in a linear space of n dimensions is equivalent to the theory of the invariants and covariants of a completely integrable system of linear partial differential equations with r independent variables, whose general solution depends on* $n + 1$ *arbitrary constants.*

If we recall our preliminary discussion regarding the arbitrariness of the space element, and the great generality which is therefore involved in the notion "r-spread in n dimensions" even as applied to ordinary space, we shall appreciate the sweeping character of this generalization which unifies such a vast domain. To the mathematician who knows that metric properties may, in a certain sense, be regarded as projective properties, it will be evident what must be added in order that this unifying principle may embrace metric geometry as well.

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ON CERTAIN NON-LINEAR INTEGRAL EQUATIONS

BY MR. H. GALAJIKIAN.

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NON-LINEAR integral equations of the Volterra type have been considered by Lalesco,* Cotton,† and Picone.‡ The two theorems of the present paper give results which are of more general character. Theorems apparently still more general have been stated very recently by Evans.§ The method used is that of successive approximations. The plan of treatment applies to integral equations of the type

* *Journal de Mathématiques*, series 6, vol. 4 (1908), p. 165; Introduction à la Théorie des Equations Intégrales, p. 127.

† *Bulletin de la Société math. de France*, vol. 38 (1910), p. 144.

‡ *Rendiconti del Circolo Matem. di Palermo*, vol. 30 (1910), p. 351.

§ Proceedings of the International Congress of Mathematicians, Cambridge, December, 1912. The present paper was completed without knowledge of Professor Evans' work, and forms one section of a Cornell University master's thesis, which was officially approved in May, 1912.