it is obvious that we have also

$$(\tau + 1)^{p^2} \equiv \tau^{p^2} + 1 \mod p^3.$$

But this congruence is implied by (7) alone, as one may readily verify by multiplying (7) by τ^{p^2}. Other cases may be dealt with similarly.

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INTEGRAL EQUATIONS.

Introduction à la Théorie des Equations intégrales. By T. LALESCO. Paris, A. Hermann et Fils, 1912. 152 pp.

L'Equation de Fredholm et ses Applications à la Physique mathématique. By H. B. HEYWOOD and M. FRECHET. Paris, A. Hermann et Fils, 1912. 165 pp.

THE theory of integral equations has been developed since the publication of Volterra's first paper in 1896, and most of the work has been done since Fredholm's fundamental memoir appeared in 1900. Yet, in this comparatively short time, the number of printed papers dealing with the subject has become so great that one approaching the subject for the first time is embarrassed by the wealth of material at his command. The two books mentioned above have been written for the beginner in the study of this interesting and useful branch of analysis. The authors have given a clear and concise exposition of the fundamental principles and of the most important results obtained up to the present time. While admitting freely that there is much yet to be done both on the theoretical side and the side of applications to mathematical physics and mechanics, there can be no doubt that the fundamental portions have already reached a form that will remain classic, and that it is now desirable to have them in book form for the convenience of the mathematical public. These two small volumes will be found very useful to the reader who wishes merely an acquaintance with the first principles of the subject, as well as to the reader who expects to attain a wider knowledge by studying the journal articles. While there is necessarily some repetition the two books may well be used together. The first one is devoted to the theory of integral equations and makes no mention of applications, while the second is concerned chiefly with the application of Fredholm's equation to the problems of mathematical physics.

The treatise of Lalesco is divided into three parts entitled: I. The Equation of Volterra, II. The Equation of Fredholm, and III. Singular Equations. Beginning in part I with the equation of Volterra of the first kind

$$\int_0^x N(x, s)\varphi(s)ds = F(x) \qquad (0 \leqq x \leqq a), \tag{1}$$

where $F(x)$ and the kernel $N(x, s)$ are known functions while φ is to be determined, the author first shows the analogy of the theory of linear integral equations to that of a system of linear algebraic equations. Throughout this part of the work it is assumed that we have to deal only with finite and integrable functions. Differentiating (1) with respect to x and assuming that $N(x, x)$ does not vanish in the interval $(0a)$ leads to an equation of the second kind, that is, of the form

$$\varphi(x) + \int_0^x N(x, s)\varphi(s)ds = F(x). \tag{2}$$

To solve this equation by successive approximations a parameter λ is introduced as a factor of the integral and the solution is assumed as a power series in λ. It is shown that the coefficients are uniquely determined and that the series represents an integral function of λ, uniformly convergent with respect to x. Furthermore it is shown that this is the only finite solution. The solution of (1) is then obtained by integration and gives Volterra's theorem:

If $N(x, y)$ and $F(x)$ are differentiable with respect to x in the interval $(0a)$ and if $N(x, x) \neq 0$, and $F(0) = 0$, the equation (1) admits a unique solution which is finite and continuous in this interval.

The restrictive conditions of the preceding theorem are not necessary for an equation of the second kind. Proceeding with the consideration of the latter type, the formulas of Volterra for the iterated and reciprocal functions are obtained by applying Dirichlet's formula to the coefficients in the series obtained above.

Section IV of the first part explains the connection between the equation of Volterra and linear differential equations. Every linear differential equation can be reduced to an equation of Volterra from which can be deduced immediately all the elementary properties of the solutions. It is interesting to note that by this process it is possible to reach the conception of a linear differential equation of infinite order. Under suitable conditions of convergence the existence and uniqueness of the solution follow.

The extension to equations involving several variables and to systems of equations offers no essential difficulties and the brief treatment merely indicates the procedure by the method of successive approximations.

The equation of Fredholm, which forms the subject of the second part of the book, differs from the equation of Volterra in that the upper limit of integration, as well as the lower limit, is constant. The problem is to find a function φ which satisfies the equation

$$\varphi(x) + \lambda \int_0^1 N(x, s)\varphi(s)ds = f(x). \tag{3}$$

The method of solution proposed by Fredholm is said to be considered by Darboux and Picard the most beautiful in analysis. In the exposition of this method the author has followed the works of Fredholm, Goursat, and Heywood.

Applying the method of successive approximations, the solution is obtained as a power series in λ, just as in the case of Volterra's equation. There is, however, an important difference. The solution of Fredholm's equation is not an integral function of λ, but is convergent if $|\lambda| < 1/N$, where N is the maximum value of $N(x, s)$ in the region considered. The solution can be written in the form

$$\varphi(x) = f(x) - \int_0^1 \mathfrak{N}(x, s, \lambda) f(s)ds,$$

where the reciprocal function (noyau resolvant)

$$\mathfrak{N}(x, y, \lambda) = N(x, y) - \lambda N_1(x, y) + \cdots + (-1)^p \lambda^p N_p(x, y) + \cdots.$$

The analytic character of the solution with respect to λ depends in the first place on that of $\mathfrak{N}$. The study of the solution is then thrown upon the study of the reciprocal function. The second member of equation (3) plays no part in this consideration.

It is shown that the reciprocal function satisfies two integral equations, one of which is

$$(4) \qquad \mathfrak{N}(x, y, \lambda) = N(x, y) - \lambda \int_0^1 N(x, s)\mathfrak{N}(s, y, \lambda)ds.$$

The analogy with a system of algebraic equations suggests that the solution of (4) should be a meromorphic function of λ, in which the denominator does not depend upon x and y. Accordingly the solution is assumed in the form

$$(5) \quad \mathfrak{N}(x, y, \lambda) = \frac{A_0(x, y) + \lambda A_1(x, y) + \cdots + \lambda^p A_p(x, y) + \cdots}{1 + a_1\lambda + \cdots + a_p\lambda^p + \cdots} = \frac{D(x, y, \lambda)}{D(\lambda)}.$$

Substituting (5) in (4) and equating coefficients of powers of λ shows that the coefficients a_p may be chosen arbitrarily. If $a_p = 0$ the series found above is obtained. In order to get the formulas of Fredholm the relation

$$(6) \qquad pa_p = \int_0^1 A_p(s, s)ds$$

is assumed. It follows then that the coefficients A_p are uniquely determined and that the functions $D(x, y, \lambda)$ and $D(\lambda)$ are integral functions of λ. This method of exposition is very neat, but appears artificial because no reason is suggested for the choice of the relation (6). The function $D(\lambda)$ is called the determinant of the kernel N and the roots of $D(\lambda) = 0$ are the characteristic values.

The second chapter of part II is devoted to a more detailed study of the reciprocal function. The first two paragraphs deal with orthogonal and biorthogonal systems of functions, followed by a treatment of the kernel of the special form

$$N(x, y) = \sum_{p=1}^{n} \varphi_p(x)\psi_p(y).$$

Returning to the general case, it is shown that if λ_1 is a characteristic value then $\lambda = \lambda_1$ is a pole of the reciprocal function. For $\lambda = \lambda_1$ the expression (5) is no longer valid. In general for $\lambda = \lambda_1$ the equation (3) does not admit a finite solution. To examine this case more in detail we consider the homogeneous equation, that is, suppose $f(x) \equiv 0$, and reach the second and third theorems of Fredholm:

II. For a characteristic value λ_1 of multiplicity n and rank (index) r, the homogeneous equation admits r linearly independent solutions called fundamental solutions. The associated equation (obtained by interchanging the variables in N) has exactly the same number of fundamental solutions.

III. The necessary and sufficient condition that the equation of Fredholm with second member $f(x)$ shall admit a solution for a characteristic value $\lambda = \lambda_1$ is that $f(x)$ shall be orthogonal to all the fundamental solutions relative to λ_1 of the associated equation.

Chapter III of the second part is entitled Special Kernels and considers principally symmetric and skew-symmetric kernels, and the development of functions in terms of fundamental functions, following the work of Hilbert and Schmidt.

The third part of the book treats of singular equations and non-linear equations. A linear integral equation is said to be singular (1) if one of the limits of integration is infinite, or (2) if the integrand becomes infinite for at least one point in the interval of integration, or (3) if, in the equation of Volterra of the first kind, $N(x, x) = 0$ for at least one point of the interval. This chapter of the theory of integral equations is not complete, and much is yet to be done towards obtaining results of a general character. The present theorems apply for the most part to special cases and show that remarkable analytic circumstances of the most diverse character may occur. For example, the characteristic values, in general, no longer form a discrete set, but may be distributed upon segments of the real axis everywhere dense, and the analytic nature of the solution in λ depends essentially on the second member of the equation.

The extensive bibliography is practically complete so far as the theory is concerned. No attempt is made to cite references to papers dealing with the applications.

Unfortunately the present edition of Lalesco's book contains many misprints which will be a source of some inconvenience to the reader.

The book by Heywood and Frechet contains three chapters. In the first is found a statement of certain problems which lead to an equation of Fredholm. They involve principally the determination of a harmonic function in a domain D of three dimensions, that is, a solution of the equation of Laplace

$$\Delta V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

which is analytic in D. Further conditions are necessary if D is infinite. The following functions are harmonic outside the attracting masses: (1) the potential of a space distribution, (2) the potential of a simple distribution on a surface, (3) the potential of a double distribution on a surface. The classic problems relative to harmonic functions are

1. *Dirichlet's Problem.*—The function V is subject to the condition of taking given values upon the surface S. $V_s = f(M)$, where M denotes a point of the surface.

2. *Neumann's Problem.*—In this case the values of the normal derivative upon S are given. $\partial V/\partial n_s = g(M)$.

3. *Problem of Heat.*—The expression $\partial V/\partial n + pV$ is given upon S, p being a function of M. The condition $q \cdot \partial V/\partial n + pV = h(M)$ reduces to the preceding if $q \neq 0$.

4. The term "mixed problem" is applied to a problem which one meets in hydromechanics and corresponds to $q = 0$ upon part of the surface and $p = 0$ upon the other part.

The problems of mathematical physics corresponding to the preceding occur in connection with the newtonian attraction and electrostatics, magnetism, hydrodynamics, elasticity, theory of heat, and acoustics. It is shown how these problems lead to integral equations of the Fredholm type.

The second chapter treats the theory of Fredholm's equation. The first solution is obtained by Neumann's method of successive substitutions and is made without the introduction of a parameter. Throughout this chapter the method of exposition and the terminology follow closely the corresponding work in Bôcher's Introduction to the Study of Integral Equations. The equation of Volterra is treated as a special case of the Fredholm equation.

In the third chapter the theory is applied to the solution of the problems stated in chapter I.

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