THE APRIL MEETING OF THE CHICAGO SECTION.

THE thirtieth regular meeting of the Chicago Section of the American Mathematical Society was held at the University of Chicago on Friday and Saturday, April 5–6, 1912, extending through four half-day sessions. The total attendance was seventy-five, including the following fifty-six members of the Society:

Mr. Charles Ammerman, Professor R. P. Baker, Professor G. N. Bauer, Dr. E. G. Bill, Professor G. A. Bliss, Dr. R. L. Börger, Professor W. H. Bussey, Professor R. D. Carmichael, Professor H. E. Cobb, Dr. A. R. Crathorne, Professor D. R. Curtiss, Professor S. C. Davisson, Mr. W. W. Denton, Dr. Arnold Dresden, Professor L. E. Dickson, Professor W. B. Ford, Dr. T. H. Gronwall, Professor A. G. Hall, Professor E. R. Hedrick, Dr. T. H. Hildebrandt, Professor O. D. Kellogg, Professor A. M. Kenyon, Professor Kurt Laves, Professor A. C. Lunn, Dr. G. F. McEwen, Professor Malcolm McNeill, Dr. W. D. MacMillan, Professor G. A. Miller, Professor E. H. Moore, Professor F. M. Morrison, Professor F. R. Moulton, Mr. E. J. Moulton, Professor Alexander Pell, Professor W. D. Pemberton, Professor H. L. Rietz, Professor W. J. Risley, Professor W. H. Roever, Miss Ida M. Schottenfels, Mr. A. R. Schweitzer, Mr. T. M. Simpson, Professor C. H. Sisam, Professor E. B. Skinner, Professor H. E. Slaught, Mr. V. M. Spunar, Dr. E. B. Stouffer, Professor E. J. Townsend, Professor A. L. Underhill, Professor E. B. Van Vleck, Dr. G. E. Wahlin, Mr. C. W. Wester, Professor J. K. Whittemore, Professor E. J. Wilczynski, Professor R. E. Wilson, Professor A. E. Young, Professor J. W. A. Young, Professor Alexander Ziwet.

Professor D. R. Curtiss, chairman of the Section, presided throughout the four sessions, except during the presentation of his own paper, when Professor E. B. Van Vleck occupied the chair.

On Friday evening about fifty members dined together in the café of the University Commons and enjoyed another of those seasons of social intercourse to which all look forward with increasing interest.

At the business meeting on Saturday morning attention was called to the fact that the next meeting of the American Association for the Advancement of Science is to be at Cleveland, Ohio, in December, 1912, and that this occasion will afford an excellent opportunity for holding a general meeting of the American Mathematical Society at the same time and place. In this connection the following resolution was unanimously adopted: That the members of the Society in attendance at the April meeting of the Chicago Section respectfully request the Council to consider the advisability of holding the next annual meeting of the Society in Cleveland in affiliation with the American Association.

Under the head of "informal notes and queries," which was introduced into the programme in order to give the members an opportunity for the interchange of mathematical suggestions not related directly to any of the papers, there was a variety of interesting discussions participated in by Professors Moore, Hedrick, Curtiss, Dickson, Baker, Bliss, Moulton, Lunn, Dr. Bill, Dr. Gronwall, and Miss Schottenfels.

The following papers were presented at this meeting:

(1) Dr. S. LEFSCHETZ: "On birational transformations of three-space related to four-space varieties."

(2) Professor W. H. ROEVER: "Optical interpretations in higher geodesy."

(3) Professor W. H. ROEVER: "Mechanisms for illustrating lines of force."

(4) Professor W. H. ROEVER: "Deviations of falling bodies for a distribution not of revolution. Second paper."

(5) Professor ARNOLD EMCH: "The geometry of conformal rational transformations in a plane."

(6) Professor KURT LAVES: "The present state of the theory of Jupiter's five minor satellites."

(7) Professor D. R. CURTISS: "An extension of Descartes' rule of signs. Second paper."

(8) Professor J. K. WHITTEMORE: "Equality in geometry."

(9) Professor G. O. JAMES: "On the relation between the empirical and the inertial trihedrons of gravitational astronomy."

(10) Professor E. J. WILCZYNSKI: "A forgotten theorem of Newton on planetary motion and an instrumental solution of Kepler's equation."

(11) Dr. G. E. WAHLIN: "The relative number field $K(\sqrt[l]{a})$."

(12) Dr. E. G. BILL: "Analytic curves in non-euclidean space. Second paper."

(13) Professor L. C. KARPINSKI: "Algebra in the Quadripartitum numerorum of Johannes de Muris."

(14) Professor G. A. MILLER: "Infinite systems of indivisible groups."

(15) Mr. W. W. DENTON: "Projective differential geometry of developable surfaces."

(16) Professor R. P. BAKER: "The method of monodromy and its application to three-parameter quartics."

(17) Professor R. D. CARMICHAEL: "On transcendentally transcendental functions."

(18) Professor R. D. CARMICHAEL: "On the theory of linear difference equations."

(19) Professor J. W. GLOVER: "A general formula for the valuation of securities."

(20) Dr. E. L. DODD: "On ordinary plane and skew curves."

(21) Professor FLORIAN CAJORI: "On the Spanish symbol U for thousands."

(22) Mr. E. W. CHITTENDEN: "Infinite developments and the composition property $(K_{12}B_1)_*$ in general analysis."

(23) Mr. V. M. SPUNAR: "Note on Mersenne's numbers."

(24) Dr. E. B. STOUFFER: "The projective differential geometry of three-spreads generated by ∞^1 planes in five-dimensional space."

(25) Professor E. H. MOORE: "Multiplicative interrelations of certain classes of sequences of positive terms."

(26) Dr. T. H. GRONWALL: "On a theorem of Fejér's and an analogon to Gibbs's phenomenon."

(27) Dr. T. H. GRONWALL: "Some asymptotic expressions in the theory of numbers."

(28) Professor F. R. MOULTON: "Closed orbits of ejection and related periodic orbits in the problem of three bodies."

(29) Dr. T. H. HILDEBRANDT: "Necessary and sufficient conditions for the interchange of limit and summation for a special type of series."

Mr. Chittenden's paper was communicated to the Society through Professor Moore. In the absence of the authors, the papers of Dr. Lefschetz, Professor Emch, Professor James, Professor Karpinski, Professor Glover, Dr. Dodd, Professor Cajori, and Mr. Spunar, were read by title. Abstracts of the papers, except that of Mr. Spunar, follow below in the order of the titles above.

1. It is known that from two depictions of a rational surface we can obtain a birational transformation between two planes, which however is only a product of several quadratic ones. The same holds true for rational four-space varieties, except that the transformations obtained are not in general the product of simpler ones. In this paper Dr. Lefschetz considers the transformations deduced from depictions of quadratic and cubic varieties. The V_3^2 give the well known quadri-quadric transformations, having for base a conic and a point in each space. Several resulting transformations are considered, the simplest being related to a V_3^3 having two conic nodes. This is obtained by projecting the first space on the variety from one of the nodes, and then reprojecting from the other on the second space; it is a cubo-cubic transformation, the fundamental elements being in both spaces a sextic intersection of a cubic and a quadric surface and a point on it. This transformation is then applied to the depiction of a sextic surface having the fundamental sextic for nodal curve and the fundamental point for quadruple point with an elliptic quartic tangent cone, which is obtained from a quadric, and has, as can also be easily verified, $p_g = p_n = P = 0$, which are the conditions of rationality of a surface as given by Castelnuovo.

2. At a point P_0 of the rotating earth, the vertical is the straight line which coincides with a plumb-line of which the bob is at P_0; the meridian plane is the plane which passes through the vertical and is parallel to the axis of rotation of the earth; the east-and-west line is the normal to the meridian plane; the latitude is the complement ϕ of the angle which the vertical (to zenith) makes with the axis (to north); the longitude is the angle λ which the meridian plane makes with a fixed plane through the axis. See Pizzetti, Geodesia Teoretica, 1905, § 5.

On a level surface, that is, a surface at each point of which the normal is the vertical, a curve of constant latitude is a locus for which ϕ is constant, for the equator $\phi = 0$; a curve of constant longitude is a locus for which λ is constant; an east-and-west curve is a locus at each point of which the tangent is the east-and-west line. In general, a tangent to a curve of constant latitude is not an east-and-west line. With the foregoing understanding of the terms involved, Professor Roever shows:

I. The curves of constant latitude are the lines of constant intensity of illumination on a level surface, with respect to a source of light infinitely far away in the direction of the axis. In particular, the equator is the line of shade.

II. The curves of constant longitude are the loci of the brilliant points of the east-and-west curves on a level surface, with respect to infinitely distant pairs of points. That is, if a level surface of the earth, supposed metallic, were scratched along its east-and-west curves, an observer on the moon, or any other distant body in any general direction, would see as a curve of light some curve of constant longitude, the curve being the assemblage of the images of the sun, which may be in any general direction, in the reflecting scratches.

Definitions of brilliant points, and photographs of the curve of light formed by the images of a source of light in the scratches which a circular saw receives during the process of polishing, are given in a paper by the author in the *Transactions*, volume 9, number 3, pages 245–279. A brass sphere, properly scratched, was exhibited in order to show the truth of the second theorem for a special case.

3. In Professor Roever's second paper the first mechanism consists of a pair of overlapping wheels, about $8\frac{1}{2}$ inches in diameter, with radial spokes. These are mounted on parallel axles and rotate in planes which are as near together as possible. These wheels can be made to rotate in the same direction or in opposite directions with equal or different angular velocities. When the wheels rotate rapidly, an observer properly situated will see a well-defined system of curves.

The second mechanism is so designed that the effect is the same as that which would be produced by the first mechanism in the limiting case in which the radius of one of the wheels becomes infinite. The author has already shown for what fields of force the systems of curves exhibited by these mechanisms are the lines of force. See *Transactions of the Academy of Science of St. Louis*, volume 7, numbers 9 and 12, and BULLETIN, volume 12, page 425.

In his Researches in Chemistry and Physics, page 292, Faraday describes the phenomenon observed in viewing one carriage wheel obliquely through another, and also that of viewing a rapidly running carriage wheel through a palisade or railing. Beyond stating that the curves in the first case resemble the lines described by iron filings between the poles of a magnet, Faraday does not identify the curves with any particular fields of force.

4. In a recent paper* Professor Roever derived an expression for the southerly deviation of falling bodies, under the assumption of a distribution of revolution, that is, that the potential function of the earth's gravitational field of force is of the form $f(r, z)$, where r is distance from the axis of rotation and z is that from a fixed plane perpendicular to the axis. He showed that for a potential function for which the standard spheroid is a level surface formula (3) below gives values nearly five times as great as the formula of Gauss,

$$2\omega^2 \sin 2\phi \frac{h^2}{6g}.$$

In the present paper he derives, under the assumption of a distribution not of revolution, the following expressions for the easterly deviation (E. D.) and the southerly deviation (S. D.):

$$(1) \quad \text{E. D.} = \frac{2}{3}\sqrt{2}\omega \cos \phi \frac{h^{\frac{3}{2}}}{\sqrt{W_z}} - \frac{5}{6}\frac{W_{yz}}{W_z} h^2,$$

$$(2) \quad \text{S. D.} = [2\omega^2 \sin 2\phi + 5W_{xz}] \frac{h^2}{6W_z} - \frac{\sqrt{2}}{15}\omega[9 \sin \phi W_{yz} + \cos \phi W_{xy}] \frac{h^{\frac{5}{2}}}{W_z^{\frac{3}{2}}},$$

where ω is the angular velocity of the earth's rotation, h is the height through which the body falls, ϕ is the latitude of the initial point P_0, W is the potential function of the relative statical field, that is, the field of force in which the plumb line is in equilibrium, x, y, z are distances measured to the north, east, and downward from P_0, and W_z, W_{xz}, W_{yz}, W_{xy} are the values of the derivatives of W, with respect to the indicated subscripts, at P_0. Hence

$$W_2 = g, \quad W_{xz} = \frac{\partial g}{\partial x}, \quad W_{yz} = \frac{\partial g}{\partial y},$$

where g is the acceleration.

* *Transactions*, vol. 12, No. 3, pp. 335–353.

For a distribution of revolution $W_{yz} = 0$, $W_{xy} = 0$; hence the second terms disappear, being negligible in comparison with the first, that is,

$$\text{S. D.} = \left(2\omega^2 \sin 2\phi + 5\frac{\partial g}{\partial x}\right)\frac{h^3}{6g}.$$

The second derivatives of W which appear in (1) and (2) can be measured experimentally by a method due to Baron Eötvös.* In a certain survey made by Baron Eötvös† values of $\partial g/\partial x$ were found more than ten times as great as the value which corresponds to the potential function which yields the standard spheroid. It is thus seen that it is possible to account for southerly deviations which are nearly fifty times as great as those given by the formula of Gauss.

5. A conformal rational algebraic transformation between two planes may be established by an irreducible rational fraction $w = f(z)/g(z)$ of the complex variable z. Writing $f(z) = u + iv$, $g(z) = r + is$, and separating real from imaginary parts,

$$w = \frac{ru + sv}{r^2 + s^2} + i\frac{rv - su}{r^2 + s^2}.$$

Professor Emch's paper deals with the properties of the system of curves $u = 0$, $v = 0$, $r = 0$, $s = 0$, $ru + sv = 0$, $rv - su = 0$, $u^2 + v^2 = 0$, $r^2 + s^2 = 0$ and their relation to the complex domain. In conclusion some geometrical applications of particular cases are given.

6. This paper forms part of an account of the theory of the satellites of the solar system prepared by Professor Laves for the sixth volume of the Encyklopädie der mathematischen Wissenschaften. It calls attention to the very important recent improvements in our knowledge of this much neglected subject. Indications are not lacking that the theory of Laplace-Souillart must in the near future be replaced by a completely new theory, which abandons the method of variation of constants and takes the equator of Jupiter as the

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* *Encyklopädie der mathematischen Wissenschaften,* Band VI, 1 B., Heft 2, §23, p. 166.

† Verhandlungen der 15. allgemeinen Conferenz der internationalen Erdmessung 1906, 1, p. 337.

fundamental plane, and not Jupiter's orbital plane, which has so far been the adopted plane of reference. It stands to reason that Professor Griffin's paper in volume 9 of the *Transactions* will form the foundation for this new theory for the satellites I, II and III.

7. In a paper read at the April, 1910, meeting of the Section, Professor Curtiss proved the theorem that for any polynomial $f(x)$ there exist polynomials $\phi(x)$ such that the number of variations of sign in the product $f(x)\phi(x)$ is exactly equal to the number of positive roots of $f(x)$. Before this paper was published, an article by Meissner appeared in the *Mathematische Annalen,* discussing the case where $f(x)$ has no positive roots, and introducing certain geometrical methods. This suggested new phases of the problem which are discussed in the present communication. A second proof of the fundamental theorem is given, based on a well-known theorem due to Laguerre. The determination of all cartesian multipliers, $\phi(x)$, of given degree r, is made to depend upon properties of a configuration of linear $(r - 1)$-spaces in space of r dimensions. Preliminary evaluations for the least degree possible for a cartesian multiplier are made, and various formulas are compared in a simple case.

8. The proofs generally given of several theorems of fundamental importance in solid geometry are based on intuitive relations assumed to exist between geometrical and physical objects, and are not admissible in a rigorous treatment of geometry. Among these are: dihedrals having equal plane angles are equal; symmetrical trihedrals are equal if isosceles, otherwise not; two prisms are equal if a trihedral of one is equal to a trihedral of the second, and if corresponding face angles of the equal trihedrals are corresponding angles of equal faces of the prisms.

Professor Whittemore shows that rigorous proofs of these and other theorems may be given on the basis of two simple assumptions and the following definition of equality of two geometrical configurations: Two configurations C and C' are equal if there is a one to one correspondence of all points A of space to all points A' of space such that (1) if A is on C, A' is on C', and conversely, if A' is on C', A is on C; (2) all relations of number and order of points A are identical with the relations of number and order of points A'.

9. In this paper Professor James, following the method of Seeliger, considers the effect of an assumed motion of the empirical trihedron of gravitational astronomy relative to the inertial on the outstanding differences between observation and theory in the motion of the four inner planets as derived by Newcomb. After correction for this motion, the effect of replacing the newtonian law of attraction by one consistent with the relativity theory is studied and the equations of motion of an infinitesimal planet about the sun

$$\frac{d^2x_i}{dt^2} = \frac{k^2M}{r^3} \cdot (- x_i + r\epsilon\xi + x_i\phi^2)\left(\frac{1}{\sqrt{1 - \phi^2}}\right)^n \quad (i = 1, 2, 3),$$

set up by Professor de Sitter, are integrated and the exponent n so determined as to remove the excess in the motion of Mercury's perihelion. It is shown that the empirical terms in the motion of the four inner planets then become less than their mean errors.

10. If the mean anomaly M, which is proportional to the time, and the radius vector r of the planet be considered as rectangular coordinates, the (M, r) curve is a trochoid. This theorem, due to Newton, seems to have been almost universally forgotten. Professor Wilczynski points out that this theorem is important in the first place as giving an intuitive notion of the functional relation between the radius vector and the time, and in the second place as offering the possibility for the construction of a simple instrument by means of which r may be obtained without any calculation whatever. Thus, the solution of Kepler's equation would become altogether unnecessary if the instrument were constructed with the requisite degree of accuracy. In all cases, however, even a moderately accurate machine of this kind would be of great assistance by giving a close approximation to the value of the eccentric anomaly E, the final value then to be obtained by the application of the usual differential formulas.

11. If we denote by k any algebraic number field, and let α be any integer in k which is not the lth power of an integer in the same field, where l is a rational odd prime, then $\sqrt[l]{\alpha}$ determines a field K of degree l relative to k. Dr. Wahlin gives in this paper a complete determination of the decom-

position of the prime ideals of k into their prime factors in K. The paper is a generalization of the one presented by the author at the April meeting, 1911, as well as of the first part of the article by Westlund in the *Transactions*, 1910. The same problem for the case $l = 2$ is solved in an article by Hilbert in the *Mathematische Annalen*, volume 51.

12. This paper is a continuation of the work which Dr. Bill presented at the December, 1911, meeting of the Chicago Section. Having classified all the analytic curves of non-euclidean space and derived the Frenet-Serret differential equations for the so-called regular curves, he comes to a discussion of the natural equations of this latter set of curves.

13. Johannes de Muris, who flourished in the first half of the fourteenth century, wrote extensively on music as well as mathematics. His "Quadripartitum numerorum" has been referred to by many writers, but only a small portion of the second book dealing with arithmetic has been published. Professor Karpinski continues his studies in medieval algebra by a discussion of the contents of the third book of the Quadripartitum, which deals with algebra. This book is entitled "Novus algorismus de additis et diminutis." The study shows that de Muris drew not only from Al-Khowarizmi, but further that he took many problems almost word for word from Leonardo of Pisa. Inserted between the third and the fourth books is a so-called Semi liber which treats of the extraction of roots and again follows Leonardo. This study is based upon the Codex Pal. Vind. (Vienna), 4770, in which the third book is found on pages 229ver.—261rec.

14. A group is said to be divisible if it is the direct product of two or more groups none of which is identity. If it is not such a direct product it is said to be indivisible or prime. If a group is divisible it can be resolved into indivisible factor groups in essentially only one way. Professor Miller considers mainly various infinite systems of indivisible groups which are such that no two groups of the system have the same order. Among the theorems established are the following: A necessary and sufficient condition that a solvable group is a direct product of a Sylow subgroup and another subgroup is that its group of inner isomorphisms involves the corresponding

Sylow subgroup as a factor of a direct product, whenever it involves such a Sylow subgroup. A group of order p^m, p being any prime number, whose central is cyclic is always indivisible. A necessary and sufficient condition that the holomorph of a cyclic group is indivisible is that the order of this cyclic group is a power of a single prime number.

15. In this paper Mr. Denton presents the details of a projective differential theory of developables, which was indicated by Professor Wilczynski at the April, 1909, meeting of the Chicago Section.*

16. Professor Baker's paper has for its aim the establishment of an effective technique. It is proved that ordinary points on the discriminantal locus give rise to simple transpositions as elements of the monodromy group. The singularities, including the intersection of the various sheets in the case of reduction, give any other types of substitution which arise. By birational transformation of the resolvent equations three parameter quartics of each of the transitive groups are made available as material. In the case of non-symmetric groups, this can be so arranged that one parameter is removable by a rational process and is ineffective for monodromy. Examples are given of all cases of irreducible three-parameter quartics with assigned group and assigned invariant subgroup as monodromy group; also of the determination of the algebraic number whose adjunction reduces the group to the monodromy group.

17. Following the classification made by Moore in his fundamental paper † on transcendentally transcendental functions Professor Carmichael calls a transcendental function y of x algebraically or transcendentally transcendental according as it does or does not satisfy an algebraic differential equation. In the first part of the paper attention is confined to functions y of x expansible in the form

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots,$$

where $a_0, a_1, a_2, \cdots$ are rational fractions in their lowest terms.

* BULLETIN, vol. 15, p. 436.

† *Math. Ann.*, vol. 48, pp. 49–74.

Among the general results obtained the following may be mentioned: (1) If y is an integral of an algebraic differential equation, then there exists an integer τ such that, for increasing n, the denominator of a_n ultimately becomes and remains less than n^{τ^n}. (2) If y is an integral of an algebraic differential equation and if the denominator of a_n is written in the form $r_n s_n$, where s_n is prime to $n!$ and to the denominator of every a_i, $i < n$, then there exists an integer ϵ such that, for increasing n, s_n ultimately becomes and remains less than $n\epsilon$. (3) Let $b_0, b_1, b_2, \cdots$, be a set of rational fractions in their lowest terms, an infinite number of which are different from zero; let them have the property that a number N exists such that, for $n > N$, neither the numerator nor the denominator of b_n contains a prime factor greater than n. Let α and β be relatively prime positive integers. Then the series

$$y = b_0 + \frac{b_1}{(\alpha - \beta)^k} x + \cdots + \frac{b_n}{(\alpha - \beta)^{k_{1n}}(\alpha^2 - \beta^2)^{k_{2n}} \cdots (\alpha^n - \beta^n)^{k_{nn}}} x^n + \cdots$$

if it converges, defines a function which is transcendentally transcendental. Here k_{ij} is an integer and $k_{nn} \geqq 1$ for an infinite number of values of n for which $b_n \neq 0$. (4) Particular examples of functions coming under this theorem are those which are analytic at zero and which satisfy the following functional equations, where q is rational and $|q| \neq 1, 0$:

$$g(qx) = (1+x)g(x), \quad g(qx) - g(x) = \frac{x}{1-x}, \quad e^x - 1, \quad \sin x, \quad \text{etc.}$$

In the second part of the paper it is shown that every analytic function integral of the equation

$$g(qx) = (1 + x)g(x), \quad |q| \neq 1, 0,$$

is transcendentally transcendental. Finally the following important result is obtained: Two essentially different classes of transcendentally transcendental functions are defined by linear difference equations with rational coefficients and by linear q-difference equations with rational coefficients.

18. Guichard, Appell, and Hurwitz have each proved by indirect means the important theorem that if $G(x)$ is any

entire function then there exists another entire function $u(x)$ verifying the relation

$$u(x + 1) - u(x) = G(x).$$

In the first part of his second paper Professor Carmichael proves this theorem by direct means; that is to say, a power series is assumed for $u(x)$ and substitution is made in the above equation. There results an infinite system of equations for determining the coefficients of this power series. This system is solved by what is believed to be a novel method of importance, and in such way that the resulting function $u(x)$ is entire.

The remainder of the paper is devoted to certain generalizations of the above result.

19. Professor Glover derives a formula for the valuation of a very general type of security. The security is redeemed in r equal instalments at intervals of t years, the first redemption being made after f years. The annual rate of dividend is g payable in m instalments and the security is purchased to realize the investor a nominal rate of interest with frequency of conversion m. The value is expressed in terms of a single function, namely, the present values of immediate annuities certain.

20. Dr. Dodd shows that the conditions of continuity, differentiability, and convexity imposed by Pringsheim * in defining an ordinary arc and its function $y = f(x)$ are not coextensive with the conditions used by Du Bois-Reymond,† who, in place of convexity, requires that there should not be an infinite number of maxima with respect to any straight line. But Pringsheim's conditions necessitate the continuity and univariance of the first derivative except at the end points of the interval, where being univariant it may become infinite. Pringsheim's ordinary arc is Osgood's ‡ regular arc defined by

$$x = \varphi(t), \quad y = \psi(t), \qquad (t_0 \leqq t \leqq t'),$$

* Encyklopädie, II, A. 1, p. 22.
† *Crelle*, vol. 79, p. 32.
‡ Lehrbuch der Funktionentheorie, p. 43.

where φ and ψ are continuous and have continuous first derivatives which do not simultaneously vanish in (t_0, t'), with the added condition that ψ'/φ' be univariant in (t_0, t').

The definition of an ordinary arc is then restated to make it invariant to rotation, a generalization for three dimensions is given, and finally straight lines, ordinary plane arcs, and ordinary skew arcs are distinguished by means of the Wronskians of the first derivatives of the defining functions, and the minors of the Wronskians.

21. Professor Cajori points out that, at the beginning of the nineteenth century and for three hundred years previous, Spanish-American manuscripts often contain, along with the Hindu-Arabic numerals, symbols taking the form of a U or V or O, or sometimes an old form of the Greek letter sigma, to designate "thousands." Thus, 7U291 = 7291.

22. In the "Introduction to general analysis" E. H. Moore has formulated a theory of developments of a general range P. A development of P is defined to be a denumerably infinite sequence of stages, each stage consisting of a finite system of subclasses of P. In terms of a development Moore and A. D. Pitcher have secured a body of theorems relating to classes of functions on a general range. Mr. Chittenden considers developments whose stages are denumerably infinite systems of subclasses and obtains theorems which include those of Moore and Pitcher and cover additional cases.

24. The projective differential geometry of ruled surfaces in ordinary space has been developed by Professor Wilczynski by means of a system of two linear homogeneous differential equations of the second order. In the present paper Dr. Stouffer develops by similar methods the projective differential geometry of three-spreads generated by ∞^1 planes in five-dimensional space by making use of a system of three linear homogeneous differential equations of the second order. The three-spreads for which any two consecutive generating planes intersect are excluded from the theory.

25. Consider sequences α: $a_1, a_2, a_3 \cdots$ of positive terms and denote by $\mathfrak{C}_u$, where u is a non-vanishing real number, the class of all sequences α whose u power sequences α^u of positive terms

have convergent sums $\mathfrak{a}_1^u + \mathfrak{a}_2^u + \mathfrak{a}_3^u + \cdots$, and by $\mathfrak{D}_u$ the class of all sequences α whose u power sequences α^u have divergent sums. Then, d, e, f being three positive numbers such that $(d + e)f = d + e$, we have the multiplication theorems

$$\mathfrak{C}_d\mathfrak{C}_e = \mathfrak{C}_f, \quad \mathfrak{C}_e\mathfrak{D}_{-f} = \mathfrak{D}_{-d}.$$

These theorems are convenient normalizations, slightly generalized, of theorems due to Pringsheim* and Landau,† which especially for $(d, e, f) = (2, 2, 1)$ play a rôle in the theory of functions of infinitely many variables.

Professor Moore determines a closed multiplication table for a set of classes of sequences of positive terms whose definitions involve, besides the notions u power and convergence, the notions finiteness and exponent of convergence.

26. Let $f(x)$ be continuous and $|f(x)| \leqq M$ for $0 \leqq x \geqq 2\pi$, and suppose that the Fourier coefficients of $f(x)$ satisfy the inequalities

$$|a_n| \leqq A/n, \quad |b_n| \leqq B/n \quad (n = 1, 2, \cdots).$$

Let

$$s_n(x) = a_0 + a_1 \cos x + b_1 \sin x + \cdots + a_n \cos nx + b_n \sin nx$$

be a partial sum of the Fourier series for $f(x)$; Fejér has shown that

$$|s_n(x)| \leqq M + \frac{1}{n+1}\left(A\sum_{v=1}^{n}|\cos vx| + B\sum_{v=1}^{n}|\sin vx|\right),$$

whence

$$|s_n(x)| < M + A + B.$$

In Dr. Gronwall's paper, this inequality is replaced by the following closer ones:

$$|s_n(x)| < M + \sqrt{A^2 + B^2}$$

for any values of A and B, and

$$|s_n(x)| < M + \tfrac{1}{2}A + \frac{\sqrt{3}}{2}B$$

for $B \geqq \sqrt{3}A$.

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* *Münchener Berichte*, vol. 32 (1902), p. 177.

† *Göttinger Nachrichten*, 1907, pp. 25–27.

Making

$$U_n(x) = \frac{1}{n+1}(|\cos x| + |\cos 2x| + \cdots + |\cos nx|),$$

$$V_n(x) = \frac{1}{n+1}(|\sin x| + |\sin 2x| + \cdots + |\sin nx|),$$

it is further shown that $\lim_{n=\infty} U_n(x)$ and $\lim_{n=\infty} V_n(x)$ are pointwise discontinuous functions of x, having the following values:
For $x = 0$,

$$\lim_{n=\infty} U_n(x) = 1, \quad \lim_{n=\infty} V_n(x) = 0;$$

for $x = \frac{p}{q}\pi$, where $p\ (\neq 0)$ and q are integers and relative primes,

$$\lim_{n=\infty} U_n(x) = \frac{1}{2q}\left(\cot\frac{\pi}{4q} + (-1)^{q+1}\tan\frac{\pi}{4q}\right),$$

$$\lim_{n=\infty} V_n(x) = \frac{1}{q}\cot\frac{\pi}{2q};$$

for $x : \pi$ irrational,

$$\lim_{n=\infty} U_n(x) = \lim_{n=\infty} V_n(x) = \frac{2}{\pi}.$$

It is finally shown that

$$\tfrac{1}{2} = \text{min.} \lim_{n=\infty} U_n(x) = \lim_{n=\infty} \text{min.}\ U_n(x) < \text{max.} \lim_{n=\infty} U_n(x) =$$
$$= \lim_{n=\infty} \text{max.}\ U_n(x) = 1,$$

$$0 = \text{min.} \lim_{n=\infty} V_n(x) = \lim_{n=\infty} \text{min.}\ V_n(x) < \text{max.} \lim_{n=\infty} V_n(x) =$$
$$= \frac{2}{\pi} < \lim_{n=\infty} \text{max.}\ V_n(x) = 0.724 \cdots,$$
$$\text{max.}\ V_n(x) < \text{max.}\ V_{n+1}(x),$$

so that $V_n(x)$ presents an analogue to Gibbs' phenomenon, while there is none in the case of $U_n(x)$.

27. Let

$$s_\alpha(x) = \sum_{d/x} d^\alpha$$

be the sum of the αth powers of all divisors of the integer x. Dr. Gronwall shows that

$$\limsup_{x=\infty} \frac{s_\alpha(x)}{x^\alpha} = \xi(\alpha) \quad \text{for } \alpha > 1,$$

$$\limsup_{x=\infty} \frac{s_\alpha(x)}{x \log\log x} = e^C \quad \text{for } \alpha = 1,$$

$$\limsup_{x=\infty} \frac{\log s_\alpha(x)/x^\alpha}{(\log x)^{1-\alpha}/\log\log x} = \frac{1}{1-\alpha} \quad \text{for } 0 < \alpha < 1,$$

where $\xi(\alpha)$ is the Riemann zeta function and C the Eulerian constant.

28. In this paper Professor Moulton proves the existence of orbits of ejection which are also orbits of collision with the same finite body. These orbits are the limits of two classes of periodic orbits.

In his Stockholm lectures Painlevé conjectured that an orbit satisfying a certain analytic condition would be an orbit of ejection or collision. In this paper it is proved that any point and an arbitrary positive velocity may be selected, and that an orbit of collision and one of ejection exist which pass through the point with the given velocity. The direction of motion is an analytic function of the coordinates of the point and the velocity.

29. The chief theorem in Dr. Hildebrandt's paper is the following: Suppose $m > 0$,

$$\sum_p |x_{np}|^m$$

convergent for every value of n, and

$$\lim_n x_{np} = x_p$$

for every value of p. Then the necessary and sufficient condition that

$$\sum_p |x_p|^m$$

be convergent and

$$\lim_n \sum_p | x_{np} |^m = \sum_p | x_p |^m$$

is that

$$\lim_n \sum_p | x_{np} - x_p |^m = 0.$$

In the proof of this theorem there arises another necessary and sufficient condition concerning itself with the limit of the sum of a sequence of series of positive terms.

H. E. SLAUGHT,
Secretary of the Section.

THE TWENTY-FIRST REGULAR MEETING OF THE SAN FRANCISCO SECTION.

THE twenty-first regular meeting of the San Francisco Section of the Society was held at Stanford University, on Saturday, April 6, 1912. About fifteen persons were present, including the following members of the Society:

Professor R. E. Allardice, Mr. B. A. Bernstein, Dr. Thomas Buck, Professor H. F. Blichfeldt, Professor G. C. Edwards, Professor M. W. Haskell, Professor L. M. Hoskins, Professor D. N. Lehmer, Professor H. C. Moreno, Professor E. W. Ponzer.

Morning and afternoon sessions were held, Professor Hoskins, chairman of the Section, presiding.

The following papers were presented at this meeting:

(1) Mr. B. A. BERNSTEIN: "On the relation between spaces in n-dimensional space and their concrete representation for the space of four dimensions."

(2) Dr. THOMAS BUCK: "Some periodic orbits of three finite bodies."

(3) Dr. S. LEFSCHETZ: "On cubic surfaces and their nodes."

(4) Professor H. F. BLICHFELDT: "On the order of linear homogeneous groups. Fifth paper."

(5) Mr. B. A. BERNSTEIN: "On an algebra of probability" (preliminary communication).

In the absence of the author the paper by Dr. Lefschetz was read by title. Abstracts of the papers follow.