$$(15)\quad \begin{aligned} s^2\sigma^2 + 2\rho b + a^2 + (s^2 + \sigma^2)\rho^2 - s\sigma(s + \sigma)2\rho \\ + s\sigma(2a - 2\rho^2) + (s + \rho)(2\rho a + 2b) = 0, \end{aligned}$$

$$(17)\quad v^2s^2 + 2avs + 2b(v + s) + a^2 = 0,$$

$$(18)\quad w^2\sigma^2 + 2aw\sigma + 2b(w + \sigma) + a^2 = 0.$$

Elimination of s from (15) and (17) gives an equation symmetrical in v and σ, which is thus identical with the result of eliminating w from (27) and (18), so that (27) is consistent with (15), (17) and (18). It may be shown by actual division that the *square* of (27) is a factor of (23). In carrying out the division, the symmetry of (27) as to ρ, v and w, in connection with the assumed symmetry of (23) in the same quantities, permits many terms of the quotient to be written by inspection after a few have been obtained. At this stage the knowledge of the factors for the special case $\rho = 0$ determines many coefficients in the general case. The entire computation has been carefully checked, and the single assumption of symmetry verified, but the length of the quotient (one hundred ninety-eight terms) prevents its reproduction in this paper. It is of particular importance that this identical transformation, when employed to obtain new curves from the $\wp$-function curves in connection with the theory of conjugate functions, reproduces the original curves.

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SURFACES IN HYPERSPACE WHICH HAVE A TANGENT LINE WITH THREE-POINT CONTACT PASSING THROUGH EACH POINT.

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THROUGH each point on a surface in ordinary space S_3 pass two tangents having with the surface three-point contact (tangents to the asymptotic lines). The osculating planes to these curves are also tangent to the surface at the point of osculation. It is easily seen that the lines on a ruled surface in hyperspace have these same properties. The question then naturally arises whether there exist in hyperspace surfaces not ruled having the same sort of lines. It is the object of this paper to show the existence of such surfaces and to derive the differential equation which the coordinates must satisfy. The results are generalized for spreads of higher dimensions than two.

1. The existence of such surfaces can be shown geometrically as follows: On a ruled surface in a space S_n trace three infinitely near curves c, c_1, c_2. Through the curves c_1 and c_2 pass a second ruled surface infinitely near the first and on this surface trace a third curve c_3 infinitely near to c_1, c_2. Then through the curves c_2, c_3 pass a third ruled surface infinitely near to the second and on it trace a curve c_4 infinitely near c_2, c_3. Continuing this process to the limit, it is at once evident that the locus of the curves c is a surface of the kind desired.

We will now derive the condition that a surface in S_n shall have through each point a curve whose osculating plane is tangent to the surface.

Let the surface be

$$x^{(i)} = x^{(i)}(u, v) \quad (i = 1, 2, 3, \cdots, n + 1) \tag{1}$$

and let three neighboring points be

$$(2) \quad x, \qquad (3) \quad x_1du + x_2dv,^*$$

$$x_{11}du^2 + 2x_{12}dudv + x_{22}dv^2 + x_1d^2u + x_2d^2v. \tag{4}$$

The tangent plane to the surface at the point x is defined by the three points†

$$x, \; x_1, \; x_2.$$

If the given curve is to have this plane for osculating plane, it is only necessary that the point

$$x_{11}du^2 + 2x_{12}dudv + x_{22}dv^2 \tag{5}$$

should lie in it; for the points (2), (3), and $x_1d^2u + x_2d^2v$ already lie in it. The condition then that (4) should lie in the tangent plane is

$$(6) \quad x_{11}du^2 + 2x_{12}dudv + x_{22}dv^2 + Ax_1 + Bx_2 + Cx = 0.$$

* Subscripts denote derivatives with respect to a variable t, where the curve on the surface is defined by assuming u and v to be functions of t. Hereafter superscripts will be omitted where no ambiguity occurs.

† See Segre, "Su una classa di superficie degl'iperspazii ecc.", *Atti di Torino*, 1907.

If the coordinates satisfy (6), then the points (2), (3), (4) will lie in the tangent plane. Hence:

If the coordinates of a surface in S_n satisfy a parabolic partial differential equation of the second order, then through each point of the surface passes a curve whose osculating plane coincides with the tangent plane to the surface at the point, and conversely.*

If now a tangent line is to have three-point contact with the surface, two conditions are necessary: (*a*) the point (5) must lie in the tangent plane; (*b*) for some choice of $d^2u : d^2v$ the point (4) must lie on the tangent line. The point (4) lies on the line joining (5) to $x_1d^2u + x_2d^2v$, and consequently if (6) is satisfied then when $d^2u : d^2v$ varies the point (4) will describe a line in the tangent plane. This line will cut the tangent line drawn in the direction du/dv. Thus if condition (*a*) is satisfied, condition (*b*) is also satisfied. Hence:

The necessary and sufficient condition that a surface in S_n have through each point a tangent line having three-point contact with the surface is that the coordinates of the surface satisfy a parabolic partial differential equation of the second order.

From the form of equation (6) it is seen that the directions of the three-point tangents are in the direction of the characteristics, since each is in the direction defined by du/dv.

In the paper referred to above Segre showed that a surface whose coordinates are particular solutions of two general partial differential equations of the second order must either be a developable or else lie in an ordinary space of three dimensions. From the above property it is evident that the surface could not be a developable if the coordinates satisfy two parabolic differential equations, for that would necessitate two principal directions (directions of three-point contact), which would be impossible. Then if such be the case, the surface must lie in an ordinary space S_3. In the pencil of partial differential equations which is defined by two general ones there are always two parabolic ones. Hence the two partial differential equations which the coordinates of a developable surface in hyperspace satisfy cannot be independent but must be such that there is only one parabolic equation belonging to the pencil defined by them.

2. The above considerations can be extended to varieties

* See Segre, loc. cit., §15.

of any number of dimensions contained in a space S_n of n dimensions. Let the coordinates of the variety V_m be

$$(7) \qquad x^{(i)} = x^{(i)}(u_1 u_2 \cdots u_m) \quad (i = 1, 2, \cdots, n + 1).$$

The tangent S_m to V_m is defined by the points

$$x, \quad x_1, \quad x_2, \quad \cdots, \quad x_m$$

and if the osculating plane to a curve traced on the variety lies in this tangent S_m, then the three points

$$(8) \quad x, \qquad (9) \quad \sum_{i=1}^{i=m} x_i du_i,$$

$$(10) \qquad \sum_{i=1}^{i=m} \sum_{j=1}^{j=m} x_{ij} du_i du_j + \sum_{i=1}^{m} x_i d^2 u_i$$

must lie in it. But as before, (8), (9), and $\sum_{i=1}^{m} x_i du_i$ already lie in the tangent S_m and therefore the only condition necessary is that the point

$$(11) \qquad \Sigma\Sigma x_{ij} du_i du_j$$

should lie in it. This condition is

$$(12) \qquad \sum_{i,j=1}^{m} x_{ij} du_i du_j + \sum_{i=1}^{m} A_i x_i + A_{n+1} x = 0.$$

It is evident, since there are $m - 1$ independent quantities $du_1 : du_2 : du_3 : \cdots : du_m$ at our disposal and $m + 1$ quantities A, that equation (12) can always be satisfied if $2m > n$. And if $2m < n$, then the coordinates x must be particular solutions of (12) if the osculating plane to a curve traced on V_m lies in the tangent S_m. In this case, as in the preceding one, it is seen that if the point (11) lies in the tangent S_m, then the ratios $d^2u_1 : d^2u_2 : \cdots : d^2u_m$ can be so determined that (10) will lie in the line joining the points (8) and (9). Hence:

If through each point of a variety V_m in S_n there is a principal direction, then the coordinates must be particular solutions of the partial differential (12). *The osculating planes of curves enveloped by these principal directions will lie in the tangent S_m to the V_m.*

From equation (12) we see that the principal directions passing through each point of a hypersurface V_{n-1} in S_n form a quadric cone of $n - 2$ dimensions, and the principal directions through each point of a variety V_{n-k} $(2k<n)$ form a cone of order 2^k and dimensions $n-2k$. If $n=2m-1$ and $k=m-1$ the number of principal directions will be equal to 2^{m-1}. The curves enveloped by these directions correspond to the asymptotic curves on a surface in S_3 and in fact can be obtained by the consideration of conjugate directions just as in S_3. Take any two infinitely near points of V_m and draw the tangent spaces S_m to V_m at each of these points. The two S_m's will intersect in a line and this line is said to be conjugate to the direction defined by the two points of tangency. Using the same method as in a previous paper* by the author, the condition that the two directions du and δu be conjugate is found to be the vanishing of the $(m+2)$-rowed determinants of the matrix

$$\| x, x_1, x_2, \cdots, x_m, \sum_i \sum_j x_{ij} du_i \delta u_j \|.$$

The condition that the two conjugate directions should coincide is seen to be that the coordinates should satisfy equation (12). In this case then we have a new property of such curves, viz.: The tangent spaces S_m to V_m at two infinitely near points of the curve enveloped by the principal tangents intersect in the line joining the two points, that is, in a principal tangent. By an easy extension of the method used by Segre† it can be shown that this is a property of such curves in general. Or it may be seen that if this property does hold in general there is an S_{m+1} having two-point contact (i. e., tangent at two infinitely near points) with the variety V_m. Now the tangent at x is determined by

$$(13) \qquad x, \quad x_1, \quad x_2, \quad x_3, \quad \cdots, \quad x_m$$

and the tangent at the point

$$(14) \qquad \Sigma x_i du_i$$

is determined by

$$(15) \qquad \Sigma x_i du_i, \quad \Sigma x_{1j} du_j, \quad \Sigma x_{2j} du_j, \quad \cdots, \quad \Sigma x_{mj} du_j.$$

It is seen at once that the tangent space (13) contains the point (14) and that the tangent space (15) contains the point

$$(11) \qquad \Sigma x_{ij} du_i du_j$$

* *Annals of Mathematics*, vol. 13, p. 89.

† Loc. cit.

Now if the space (13) contains (11), the two tangent spaces will have the line defined by the direction $du_1: du_2: \cdots : du_m$ in common. This is exactly condition (12). Hence:

The curves enveloped by the principal directions on a variety whose coordinates satisfy (12) *are such that the tangent S_m's at two consecutive points intersect in the line joining the two points.*

3. Let us now find the condition that a variety V_3 have through each point a tangent with four point contact with V_3. The condition may be got by first finding the condition that curves exist on V_3 whose osculating S_3 at any point coincides with the tangent S_3 to V_3 at that point. This requires that the two points

$$(16) \qquad \Sigma x_{ij} du_i du_j + \Sigma x_i d^2 u_i$$

$$(17) \qquad \Sigma x_{ijk} du_i du_j du_k + 3\Sigma x_{ij} du_i d^2 u_j + \Sigma x_i d^3 u_i$$

should lie in the S_3 determined by the points

$$x, \quad x_1, \quad x_2, \quad x_3.$$

This is equivalent to the relations

$$(12) \qquad A\Sigma x_{ij} du_i du_j + \sum_1^3 B_i x_i du_i + Cx = 0,$$

$$(18) \quad A'(\Sigma x_{ijk} du_i du_j du_k + \Sigma x_{ij} du_i d^2 u_j) + \Sigma B_i' x_i du_i + C'x = 0.$$

In relation (12) the du_i define the direction of the curve. If (12) is satisfied, then the determinants of the matrix

$$\|\Sigma x_{ij} du_i du_j, x, x_1, x_2, x_3\|$$

must vanish. Now if we change du_i into $du_i + \frac{1}{2} d^2 u_i$, this relation must still be satisfied; this leads to

$$\| \Sigma x_{ij} du_i d^2 u_j, x, x_1, x_2, x_3 \| = 0,$$

which used in (18) reduces that relation to

$$(19) \qquad A'\Sigma x_{ijk} du_i du_j du_k + \sum_1^3 B_i' x_i + C'x = 0.$$

Hence:

If a variety V_3 is covered with a family of curves whose osculating S_3 at each point coincides with the tangent S_3 to V_3 at that point, then the coordinates must be solutions of two simultaneous partial differential equations, one of parabolic type of *the second order and the other of the third order having coincident roots of its characteristic equation. The curves in question are the characteristics.*

If equation (12) is satisfied, we saw that the tangents in the direction du_i have three-point contact with V_3. If in addition (19) is also satisfied, since (17) lies in the tangent S_3, the quantities d^3u_i can be so chosen that (17) will lie on the line joining x to $x + dx$. Hence:

When the coordinates of a spread V_3 satisfy the differential equations (12) *and* (19), *through each point of the spread passes a tangent having four-point contact with the spread.*

In the same manner it can be shown that the necessary and sufficient condition that a spread V_m have in each point a tangent with r-point contact is that the coordinates should satisfy the $r - 2$ equations

$$(20)\quad \begin{array}{l} A\Sigma \dfrac{\partial^{r-1}x}{\partial u_i \cdots \partial u_{r-1}} du_1 du_2 \cdots du_{r-1} + \Sigma B_i x_i + Cx = 0, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ A^{(r-2)}\Sigma \dfrac{\partial^2 x}{\partial u_i \partial u_j} du_i du_j + \Sigma B_i^{(r-2)} x_i + c^{r-2}x = 0. \end{array}$$

The osculating spaces to such curves always lie in the tangent S_m to V_m. In particular when $r - 1 = m$ and the coordinates satisfy the system of equation (20), there is a curve passing through each point whose tangent lines have $(m + 1)$-point contact with V_m. The osculating S_m of the curve coincides with the tangent S_m to V_m.

That equations (20) be satisfied requires $(n - m)(r - 2)$ conditions. There are at our disposal $m - 1$ ratios $du_1 : du_2 : \cdots : du_m$; therefore if the above number is less than or equal to $m - 1$, the condition is always fulfilled. In case

$$(21)\qquad (n - m)(r - 2) = m - 1$$

the number of curves through each point is finite. When $n = 2k - 1$, $m = k$, $r = 3$ the number of three point tangents passing through each point is 2^k.

From (21) can be calculated the dimensions of V_m such that it have a finite number of tangents through each point having r-point contact. In particular if the spread is to have a 4-point tangent through each point, $m = \frac{1}{3}(2n + 1)$.

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