equation $g(x+1) = xg(x)$ which have the properties that

$$\Gamma(1) = 1, \qquad \bar{\Gamma}(x) = (1 - e^{2\pi x\sqrt{-1}})\Gamma(x),$$

and

$$\lim_{x=+\infty} \Gamma(x)x^{-x+\frac{1}{2}}e^{x} \quad \text{exists.}$$

If one starts from these definitions and makes use of the general theory of linear homogeneous difference equations of the first order, the fundamental properties of $\Gamma(x)$ and $\bar{\Gamma}(x)$ are readily obtained. The theory, as worked out recently by Professor Carmichael in his academic lectures, is decidedly simpler and more elegant than the usual theory of the gamma function, as developed, for instance, in Nielsen's Handbuch.

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AN IDENTICAL TRANSFORMATION OF THE ELLIPTIC ELEMENT IN THE WEIERSTRASS FORM.

BY PROFESSOR F. H. SAFFORD.

(Read before the American Mathematical Society, April 29, 1911.)

THIS paper is based upon a formula published in 1865 in a pamphlet entitled "Problemata quaedam mechanica functionum ellipticarum ope soluta.—Dissertatio inauguralis," by G. G. A. Biermann (Berolini), where it is quoted as derived from Weierstrass's lectures. The formula is, after correcting slight misprints in Biermann's pamphlet,

$$(1) \quad F(x) = x_0 + \frac{\sqrt{R(x_0)}\sqrt{S} + \frac{1}{2}R'(x_0)[s - \frac{1}{24}R''(x_0)] + \frac{1}{24}R(x_0)R'''(x_0)}{2[s - \frac{1}{24}R''(x_0)]^2 - \frac{1}{2}A \cdot R(x_0)}.$$

F is the solution of

$$(2) \quad (F')^2 = AF^4 + 4BF^3 + 6CF^2 + 4B'F + A' = R(F).$$

The accents used with F and R denote differentiation, x_0 is an arbitrary constant, and A, B, C, B', A' are constant coefficients. Also

$$S = 4s^3 - g_2 s - g_3 = 4(s - \epsilon_1)(s - \epsilon_2)(s - \epsilon_3),$$

(3) $$s = \wp(x), \quad g_2 = AA' + 3C^2 - 4BB',$$

$$g_3 = ACA' + 2BCB' - AB'^2 - A'B^2 - C^3.$$

In Enneper, Elliptische Functionen (Halle, 1890), pages 27–30, may be found considerable discussion of the preceding formulas, while on page 59 another form of (1) is used in obtaining the addition theorem for $\wp(u)$.

Greenhill, Elliptic Functions (London, 1892), page 151, mentions (1) in connection with the reduction of the general elliptic element $dx/\sqrt{X}$ to Weierstrass's canonical form $ds/\sqrt{S}$. Haentzschel's use of Weierstrass's formula will be considered later.

From (1) an identical transformation of $ds/\sqrt{S}$ may be obtained as follows: By a proper choice of constants α, β, γ, δ, the linear transformation

(4) $$F = \frac{\alpha v + \beta}{\gamma v + \delta}$$

changes (2) into

(5) $$(v')^2 = 4v^3 - g_2 v - g_3,$$

whence, writing

(6) $$\bar{F} - x_0 = v,$$

(5) becomes

(7) $$(\bar{F}')^2 = 4(\bar{F} - x_0)^3 - g_2(\bar{F} - x_0) - g_3 = \bar{R}(\bar{F}).$$

Hence

$$\bar{R}(x_0) = -g_3, \quad \bar{R}'(x_0) = -g_2,$$

(8) $$\bar{R}''(x_0) = 0, \quad \bar{R}'''(x_0) = 4! \quad \bar{R}''''(x_0) = 0.$$

Then using the form $\bar{R}$ as indicated in (7) for R in the fundamental formula (1) gives

(9) $$v = \frac{\sqrt{(-g_3)}\sqrt{(4s^3 - g_2 s - g_3)} - \frac{1}{2}g_2 s - g_3}{2s^2}.$$

When (9) is rationalized and simplified by writing $g_2 = 4a$, $g_3 = 2b$, it gives

(10) $$v^2 s^2 + 2avs + 2b(v + s) + a^2 = 0.$$

Referring to (5), it appears that the general biquadratic X or $R(F)$ has now, by the changes above, become the same as in Weierstrass's elliptic element, so that the transformation (10) changes $ds/\sqrt{S}$ into itself, i. e., is an identical transformation, a fact which may be easily verified.

Equation (10) is obviously symmetrical in v and s, while (9) shows that the transformation is irrational in both directions. The curve corresponding to (10) has no node, except when $g_2^3 = 27g_3^2$, but consists of two distinct branches, asymptotic to both axes. Evidently the line $v - s = 0$ is an axis of symmetry and it may be shown to intersect the curve twice or not at all, according as $g_2^3 - 27g_3^2$ is negative or positive, this expression being also the criterion for the two types of $\wp$-function curves, the basis of Haentzschel's discussion.

Haentzschel, in his Reduction der Potentialgleichung (Berlin, 1893), wished to obtain a pair of families of orthogonal curves by the aid of conjugate functions. To this end he wrote

$$(11) \qquad s = \wp(t + iu), \quad \sigma = \wp(t - iu),$$

$$(12) \qquad x + iy = F(t + iu), \quad x - iy = F_1(t - iu),$$

in which F and its conjugate F_1 were defined by (2), and x_0 was complex. The orthogonal families came from the elimination of u or t respectively from (12) and writing either ρ or ρ_1 as a parameter, where

$$(13) \qquad \rho = \wp(2t) \quad \rho_1 = \wp(2iu).$$

He treated in detail the cases in which x_0 is any one of the roots of $R(x) = 0$, equation (2).

In the *Archiv der Mathematik und Physik*, series 3, volume 10, Hefte 3/4, pages 234–237, the writer has discussed some of Haentzschel's results and referred to other articles on the same point. On pages 33–37 (Potentialgleichung), Haentzschel treated the general case, i. e., x_0 unrestricted. This depends upon the $\wp$-function curves obtained from

$$(14) \qquad \begin{aligned} x + iy &= s = \wp(t + iu), \\ x - iy &= \sigma = \wp(t - iu) \end{aligned}$$

[ρ and ρ_1 as in (13), $a = g_2/4$, $b = g_3/2$], viz.,

$$(15) \quad s^2\sigma^2 + 2\rho b + a^2 + (s^2 + \sigma^2)\rho^2 - s\sigma(s + \sigma)2\rho + s\sigma(2a - 2\rho^2) + (s + \sigma)(2\rho a + 2b) = 0.$$

Haentzschel's work upon this topic concluded with an outline of the elimination of s and σ between (15) and two equations analogous to (10).

In the BULLETIN for June, 1899, pages 431–437, the writer has discussed the case in which

$$R(F) = A[\pm(F-x_0)^4 \pm c(F-x_0)^2 \pm 1], \tag{16}$$

and has factored the result, obtaining four curves each of the fourth degree.

The remainder of this paper is to be devoted to the case in which

$$\bar{R}(\bar{F}) = 4(\bar{F}-x_0)^3 - g_2(\bar{F}-x_0) - g_3. \tag{7}$$

First s and σ are to be eliminated from (15) and (10) together with the conjugate of the latter, i. e., from

$$v^2s^2 + 2avs + 2b(v+s) + a^2 = 0, \tag{17}$$

$$w^2\sigma^2 + 2aw\sigma + 2b(w+\sigma) + a^2 = 0, \tag{18}$$

$$[v = x+iy, \quad w = x-iy]. \tag{19}$$

In conclusion the resulting equation will be resolved into three components, of which two are the original $\wp$-function curves. Following Haentzschel, radicals may be avoided by writing (17) thus:

$$s^2 = -\frac{2s(av+b) + 2bv + a^2,}{v^2}, \tag{20}$$

with a similar value for σ^2.

Substituting these values of s^2 and σ^2 in (15) gives

$$Ds\sigma + Ps + Q\sigma + E = 0, \tag{21}$$

$$\begin{aligned}
\tfrac{1}{2}D &= av^2w^2 - \rho^2v^2w^2 + 2\rho av^2w + 2\rho avw^2 + 2a^2vw + 2\rho bv^2 \\
&\quad + 2\rho bw^2 + 2abv + 2abw + 2b^2, \\
\tfrac{1}{2}P &= \rho av^2w^2 + bv^2w^2 + 2\rho bv^2w - \rho^2avw^2 + 2abvw + \rho a^2v^2 \\
&\quad - \rho^2bw^2 + a^3v + 2b^2w + a^2b, \\
\tfrac{1}{2}Q &= \rho av^2w^2 + bv^2w^2 + 2\rho bvw^2 - \rho^2av^2w + 2abvw + \rho a^2w \\
&\quad - \rho^2bv^2 + a^3w + 2b^2v + a^2b, \\
E &= 2\rho bv^2w^2 + a^2v^2w^2 - 2\rho^2bvw^2 - 2\rho^2bv^2w + 4b^2vw - \rho^2a^2v^2 \\
&\quad - \rho^2a^2w^2 + 2a^2bv + 2a^2bw + a^4.
\end{aligned} \tag{22}$$

Solving (21) for s and substituting in (17) gives a quadratic in σ, between which and (18), another quadratic, σ is to be eliminated. The result is

$$(23)\quad L^2 - b^2(PQ - DE)^2(4v^3 - 4av - 2b)(4w^3 - 4aw - 2b) = 0,$$

$$(24)\quad \begin{aligned} L = {} & v^2w^2E^2 - 2(PQ - DE)(av + b)(aw + b) \\ & + Pw^2[P(a^2 + 2bv) - 2E(av + b)] + Qv^2[Q(a^2 + 2bw) \\ & - 2E(aw + b)] + [D(a^2 + 2bv) - 2Q(av + b)][D(a^2 \\ & + 2bw) - 2P(aw + b)]. \end{aligned}$$

As would be expected, (21), (22), and (23) are symmetrical in v and w, while $(v - \epsilon_1)(v - \epsilon_2)(v - \epsilon_3)(w - \epsilon_1)(w - \epsilon_2)(w - \epsilon_3)$ from the last two factors of (23) may be used to find the foci of the curves represented. Each term in (24), on reduction, is found to contain the factor v^2w^2, and this being true of $PQ - DE$ which occurs also in (23), it is possible to cancel the factor v^4w^4 from the latter and at last obtain in v and w coordinates the equation of either of the pair of orthogonal curves. The factoring and consequent resolution of (23) is much facilitated by first treating the simple case for which $\rho = 0$. The degenerate form of (15) is

$$(25)\quad s^2\sigma^2 + 2as\sigma + 2b(s + \sigma) + a^2 = 0,$$

which is of the same type as (17) and (18). Elimination of s and σ from these three gives

$$(26)\quad [v^2w^2 + 2avw + 2b(v + w) + a^2]^3[\varphi(v, w)] = 0.$$

The last factor of (26) is the form which (15) becomes when $\rho = -4(2b^2 - a^3)/a^4$, $s = v$, $\sigma = w$. But the triple factor in (26) is not preserved when the general case $\rho \neq 0$ is treated. However (15), which it will be remembered is the equation of the $\wp$-function curves, has the remarkable property of being symmetrical in all *three* of the quantities ρ, s and σ.

By reasoning which will presently be given, it may be inferred that one factor of (23) is the first member of

$$(27)\quad \begin{aligned} & v^2w^2 + 2\rho b + a^2 + (v^2 + w^2)\rho^2 - vw(v + w)2\rho \\ & \qquad + vw(2a - 2\rho^2) + (v + w)(2\rho a + 2b) = 0. \end{aligned}$$

Now (23) is the result as stated of eliminating s and σ from the following:

$$(15)\quad \begin{aligned} s^2\sigma^2 + 2\rho b + a^2 + (s^2 + \sigma^2)\rho^2 - s\sigma(s + \sigma)2\rho \\ + s\sigma(2a - 2\rho^2) + (s + \rho)(2\rho a + 2b) = 0, \end{aligned}$$

$$(17)\quad v^2s^2 + 2avs + 2b(v + s) + a^2 = 0,$$

$$(18)\quad w^2\sigma^2 + 2aw\sigma + 2b(w + \sigma) + a^2 = 0.$$

Elimination of s from (15) and (17) gives an equation symmetrical in v and σ, which is thus identical with the result of eliminating w from (27) and (18), so that (27) is consistent with (15), (17) and (18). It may be shown by actual division that the *square* of (27) is a factor of (23). In carrying out the division, the symmetry of (27) as to ρ, v and w, in connection with the assumed symmetry of (23) in the same quantities, permits many terms of the quotient to be written by inspection after a few have been obtained. At this stage the knowledge of the factors for the special case $\rho = 0$ determines many coefficients in the general case. The entire computation has been carefully checked, and the single assumption of symmetry verified, but the length of the quotient (one hundred ninety-eight terms) prevents its reproduction in this paper. It is of particular importance that this identical transformation, when employed to obtain new curves from the $\wp$-function curves in connection with the theory of conjugate functions, reproduces the original curves.

UNIVERSITY OF PENNSYLVANIA,
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SURFACES IN HYPERSPACE WHICH HAVE A TANGENT LINE WITH THREE-POINT CONTACT PASSING THROUGH EACH POINT.

BY PROFESSOR C. L. E. MOORE.

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THROUGH each point on a surface in ordinary space S_3 pass two tangents having with the surface three-point contact (tangents to the asymptotic lines). The osculating planes to these curves are also tangent to the surface at the point of osculation. It is easily seen that the lines on a ruled surface in hyperspace have these same properties. The question