gent on the interval $x - \delta$, $x + \delta$, or on that part of such interval which is an ab. In this manner for every point x_i on ab we obtain an interval of which x_i is an internal point (unless x_i is a or b, in which case x_i is the end point of such interval) on which $\sum_{n=0}^{\infty} U_n(x)$ is uniformly convergent.

By the Heine-Borel theorem* there is a finite subset of these intervals which completely covers the interval ab. Hence it follows by obvious considerations that $\sum_{n=0}^{\infty} U_n(x)$ is uniformly continuous on the interval ab.

COLUMBIA UNIVERSITY,
February 1, 1911.

PROJECTIVE GEOMETRY.

Projective Geometry. By OSWALD VEBLEN and JOHN WESLEY YOUNG. Volume I. Ginn and Company, Boston, 1910. x + 342 pp.

IN discussing this book, two facts must be constantly borne in mind: the authors knew exactly what they wanted to do, and they were entirely competent to do it. Here is no question of a slovenly work, compiled with no visible object, differing from an honorable line of progenitors only in the smallest details. The book is eminently one with a consistent purpose. Agree with the authors you may not; but if you do not fully enter into their point of view, you will come off second best in the argument.

The authors' main object is to build up projective geometry upon a system of independent axioms. Such a task is, certainly, not new. The names of Pieri, Vahlen, Schur, and others suggest themselves immediately when the question of projective assumptions is raised. Nor is there even any novelty in writing a students' textbook which starts from the ground. We have such a book already in the beautiful treatise of Enriques, and, in fact, we may almost say that this is the

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* This theorem may be stated as follows: If on an interval ab there is a set of intervals $[\sigma]$ such that (1) the points a and b are end points of intervals of the set $[\sigma]$; (2) every point of the interval ab lies within at least one interval of $[\sigma]$; then there is a finite subset $\sigma_1, \ldots, \sigma_k$ of the set $[\sigma]$ which satisfy (1) and (2) of the hypothesis.

standard way to write a textbook of projective geometry today. No, the novelty consists in the set of axioms which the authors have used. These are equally serviceable for a geometry of a finite number of points, a rational geometry, a three-dimensional continuum, or a complex continuum of three dimensions, and doubtless other forms of continua and discontinua.

The generality and high scientific interest of such a procedure are immediately evident. Heretofore the usual way has been to set up a system of assumptions sufficient and suitable for the real domain. So long a time elapses between the axiom stage and that where complex quantities are introduced that it does not occur to the student to ask whether the introduction of the new elements calls for a revision of the postulates or not. For instance, we assume that on any straight line we may find two distinct points whose distances from a given point are congruent to the distance of any two distinct points AB.* What happens in the imaginary domain when AB is null?

All such mishaps are avoided by the method adopted by our authors. There is nothing said which must later be unsaid, no assumption made which must later be discarded. At the beginning of each chapter we learn exactly what assumptions are needed for all of the theorems in that chapter; one or another assumption may not be needed, none is ever contradicted.

Such a proceeding is certainly attractive. With a minimum of assumptions, we have a maximum of generality. The complex domain is not a spirit world in which dwells the ghost that accompanies Steiner's quartic surface, but a larger ambient in which the real is immersed. But in mathematics, as everywhere else, the law of compensation holds. One can not obtain great results except at a great price, and the authors have paid a great price for the advantages obtained: they have sacrificed all that is specially characteristic of the real domain. Opinions will differ on the wisdom of such a bargain; in our judgment they have secured their gain at too great cost. Involutions appear on page 102, but from cover to cover no word as to the distinction between elliptic and hyperbolic. From page 109 onward the conic is continually in evidence, the words ellipse, hyperbola, parabola never occur. There is no way of knowing whether a coplanar line and conic intersect or not. Scientifically all this is a trivial matter, in higher mathematics the

* Hilbert's axiom, IV, 1.

complex is on all fours with the real; but didactically it is momentous. The geometry with which we begin life, the geometry of intuition, the geometry of Euclid, is the geometry of the real continuum; it takes years of arduous training before the student can really think in any other terms, so long, that is, as he is doing geometry, and not merely formal algebra. Is it wise so far to discard psychology and historical tradition in founding projective geometry?

The loss of the real domain is most keenly felt in the exercises, as there is no possibility of fruitful application of the methods developed to metrical theorems. Who that has taught projective geometry does not know what a sense of satisfaction comes over a class when they find a metrical theorem coming out of a projective one? They feel as if something familiar and concrete, almost practical, were emerging from that which had previously been abstract and remote. The student will be interested in the theorem (page 119) that if the vertices of two triangles lie on a conic, their sides touch another conic, and conversely. But he will be much more interested to find that if a triangle be circumscribed to a parabola, the circumscribed circle of the triangle passes through the focus of the parabola and, hence, that the circles circumscribing the four triangles formed by four lines are concurrent. A wide-awake student will be struck with the generality of Desargues' theorem (page 127), or its dual, whereby the pairs of tangents from any point to conics touching four lines, form pairs of an involution. Still more will he be pleased when he sees that the whole theory of confocal conics is deducible from this, or learns the Gauss-Bodenmiller theorem whereby the circles on the diagonals of a complete quadrilateral as diameters are coaxal. All of the best recent text books of projective geometry, Reye, Cremona, Russell, Enriques, Böger, agree on this didactic principle; even our authors go so far as to say (page iv) that the teacher who wishes for metrical applications may pass directly from Chapter VIII to the second volume. But the student who has mastered 235 pages of this volume will stand far less in need of metrical applications than his less fortunate brother who is at the beginning.

One other ground for postponing continuity and the real domain is offered by the authors (page iii) where they suggest that the study is too difficult and delicate for the beginner. "It will be found that the theorems selected on this basis of logical simplicity, are also elementary in the sense of being easily comprehended, and often used." Very well, how do we prove that a straight line and a conic in the same plane must intersect? We need proposition K_2, which is made to depend on K_n (page 255) proved at the end of Chapter IX: "*If* $a_x^h, a_x^l, \cdots$ *are a finite number of binary homogeneous forms, whose coefficients are proper in a space* S *which satisfies assumptions* AEP, *there exists a space* S', *of which* S *is a sub-space, in the number system of which each of these forms is a product of linear factors.*"

We are not familiar with any treatment of continuity (not even the Heine-Borel theorem) whose essence seems to us so difficult to grasp as this. Moreover, we think that the question of the difficulty of the principle of continuity is a little beside the point. Students who take up the book before us are, presumably, just at that stage where a sound grasp of this principle is vital to further progress. We might well write over the doors of our graduate class-rooms for mathematics, a paraphrase of the motto which Plato is reputed to have placed over the door of the Academy

"Let no one ignorant of continuity presume to enter here."

We must not, of course, object if someone write below the equally famous

"Lasciate ogni speranza, voi che entrate."

No, we differ fundamentally from the present authors. We believe that the starting point of geometry, scientifically and didactically, should be the real continuum. So Enriques has started in the text-book already quoted. The complex domain rests on a convenient fiction whereby we employ the terminology of n dimensions to the study of $2n$ dimensions. A thorough study of continuity, which can be admirably given in projective geometry by starting from the idea of separation, is of primary importance to every graduate mathematical student, and should on no account be postponed.

There is a second general feature of the present book on which we dwell with unreserved satisfaction, the introduction of both algebraic and geometric methods. In the past there has been a very marked cleavage between two different schools of projective geometers. On the one hand we have the disciples of Chasles, Steiner, and von Staudt, purists, who treat an algebraic proof as an unseemly intrusion into the beautiful mansion of geometry. Their position was well expressed by the late Mr. Lachlan (Preface to his Modern Pure Geometry):

"In fact it might well be taken as an axiom, based on experience, that every geometric theorem admits of a simple and direct proof, based on the principles of pure geometry." Over against these we have the school of formalists, now happily dwindling, who burn incense to syzygies and symbolic identities, regardless of how much or little geometric interest they may have. The advanced student may freely enroll himself in either of these schools, if he so choose; the beginner should have both the algebraic and the geometric point of view continually before him. We are not familiar with any other book on projective geometry (not based on metrical assumptions) which has attempted to meet this important need. We only regret that the authors have not found it possible to introduce algebraic methods earlier into their scheme, and to carry the geometric and algebraic treatment of conics, collineations, etc., forward together. Let us mention also in this connection that the book has a greater wealth of suitable exercises for the student than any other with which we are familiar, with the single exception of Russell's Pure Geometry.

One or two more general remarks. The style is, for the most part, fresh and vigorous. Even Jove will nod. On page 36 the authors take an absolutely indecent pleasure in projecting points and lines *onto* planes; on page 224 we learn that AA' *is* a conjugate pair. The general effect is clear and agreeable. The figures, on the other hand seem to us occasionally messy and unattractive. Such a figure as No. 19—two quadrangles whose corresponding pairs of sides are concurrent on a given line—suffers not a little by comparison with the corresponding figure No. 15 in Reye's Geometrie der Lage. Otherwise, the book is singularly attractive in appearance, and free from typographical errors.

Our book starts off with an introductory chapter dealing with general principles concerning the use of undefined elements, unproved propositions, etc. Simple examples are given of axiom systems which are seen to be consistent and independent. The authors know very well what they are about, and the whole discussion is luminous. Unfortunately the chapter is marred by a slip, the worst in the book. In attempting to show how the point, line, and plane at infinity come in naturally to complete our geometry, the usual cartesian coördinates are mentioned. We then learn (page 9) "If the numbers are real numbers, we are dealing with the ordinary "real" space; if they are complex numbers, we are dealing with the ordinary "complex" space of three dimensions. The following discussion applies to either case." Then, on the following page, "Now the points (x, y, z) with the exception of $(0, 0, 0)$ may also be denoted by the direction cosines of the line joining the point to the origin of coordinates, and by the distance of the point from the origin, say by $(l, m, n, 1/d)$." This statement is neither grammatically correct nor mathematically accurate, for there are ∞^2 points for which $d = 0$. A reader finding such a mistake at the outset might well expect a looseness of statement throughout, and be unfairly prejudiced against the book in consequence.

In Chapter I we make a fair start into the theorems of alignment, and the principle of duality. The fundamental objects are *points* and classes of points called *lines*. We have the necessary assumptions and existence theorems, the plane is built up on a triangular frame, the space on a tetrahedral one. It is a little hard at first to become used to the idea of a point being *on* a plane, while the plane is, at the same time, *on* the point. The use of this locution (for which the authors give the credit to Professor Morley) is amply justified however, for it enables them to make the best statement of the principle of duality with which we are familiar. The chapter ends with an excellent system of assumptions for the projective geometry of n dimensions,—or rather should end there. Actually the last thing is this exercise: "State the assumptions of extension by which to replace E_n and E_n' for a space of an infinite number of dimensions. Make use of transfinite numbers." We are told in a note that exercises such as this, which are marked with an asterisk, are of a more advanced or difficult character. We are glad to note very few such exercises in the book.

Chapter II continues the same order of ideas as are introduced in Chapter I, and is largely devoted to configurations. Desargues's theorem on perspective triangles is, of course, fundamental for this. A "configuration," by the way, is an assemblage of points, lines, and planes, wherein each element of one sort is on a constant number of elements of each of the other sorts. These numbers may be arranged in a square matrix, which is taken to symbolize the configuration. We have an impression that many of the results of this chapter, which would be fundamental in the study of finite collineation groups may appear a little dull to the beginner, but we speak here from personal prejudice rather than from settled conviction. On page 45 is a little axiom which may strike the reader as suprising. "*The diagonal points of a complete quadrangle are not collinear.*" One has an instinctive idea that such a theorem is capable of proof by means of the other assumptions, but nothing is less reliable than our instinctive ideas on such matters. A rigorous proof may be found, based on the concept of separation.* A good part of the present book is independent of this assumption.

The third chapter is given to the study of projectivities. The authors have here cut loose from the von Staudt school where the quadrangle theorem is immediately followed by the study of harmonic separation. They, like Cremona and Böger, but unlike Reye and Enriques, define as projective two one-dimensional fundamental forms which may be connected by a finite number of projections and sections. The von Staudt definition whereby projectivity depends upon one to one correspondence, with a correspondence of harmonic elements is equivalent to this in the real domain, but not in the complex one (as the authors very well know), and so finds no place. So too, in two dimensions they distinguish between a collineation between two planes, a one to one correspondence point to point, and point range to point range, and a projective collineation, where to the above is added the requirement that corresponding point ranges shall be projective. Once more in the real domain collineation and projective collineation are identical, not in the complex one. The classic example is the transformation (which the authors would call a non-projective collineation but is more commonly known as an anticollineation), where each point is carried into its conjugate imaginary. Another commendable feature of the present chapter is that after a short discussion of the abstract concept of a group, the authors take up with some care the group of one-dimensional projectivities. It is not a little curious that previous writers of text-books on projective geometry have so calmly neglected this important topic.

Chapter IV is a natural continuation of Chapter III and opens with a discussion of harmonic sets. A new and important concept is introduced in the idea of *nets of rationality*. The assemblage of all points collinear with three given collinear points, and obtainable from them by a finite number of harmonic constructions, is called the net of rationality of the first three. We have corresponding point assemblages in two and three dimensions, the discussion of the latter (pages 89–93) seems to us

* Conf. Enriques, "Geometria Proiettiva," p. 59.

somewhat unnecessarily difficult. The underlying idea is important, and is an excellent innovation in the present book. This same chapter leads us also to the fundamental theorem of projective geometry. Suppose that we have two projective one-dimensional fundamental forms (point ranges, linear pencils, etc.) with three elements self-corresponding, every element of the net of rationality of those three is self-corresponding, as may be immediately shown. The fundamental theorem (here assumption P) tells us that *every* element is self-corresponding. There are proofs in plenty of this theorem (Vahlen, Enriques, etc.) based on axioms of continuity, but the present authors have no such axiom in this first volume, and they have evolved beyond the intuitive proofs which satisfy Reye and Böger, so they adopt the natural alternative of taking it as an unproved assumption. The chapter closes with a short account of involutions, and a classification of two-dimensional collineations, based upon the discussion of the fixed elements.

Chapter V introduces the point conic, the line conic, and the conic "überhaupt" which is conceived as a one-parameter assemblage of point-line unions. The treatment follows somewhat closely that of Reye. Conics are first defined by means of projective pencils and ranges; Pascal and Brianchon then enter hand in hand, and furnish the basis for the study of the polar system. It is interesting to compare this, the Steiner method, with that of von Staudt which consists in beginning with the polar system, and defining the conic by means of the points which lie on their polar lines. The question as to which method is finally the best is not capable of categorical answer. The Steiner method gives at once a way of constructing as many points of a conic as we please. Moreover the conic is probably the most interesting figure of all to the beginner, and Steiner leads us to it in the shortest time. Our authors reach the conic on page 108, Reye, who does not bother his head about axioms, brings conics in on page 66; in Enriques, the best book on the von Staudt plan, 202 long pages must pass before conics are reached. On the other hand by the von Staudt definition conic and polar system come in together, so that the polar properties are an immediate consequence of the definition; the same is true of the equation of the conic, which may be written immediately from that of the correlation, and the procedure works equally easily in any number of dimensions. However, we digress. The comparison of these two methods, interesting as it may be in itself, has no particular bearing on the present book, at least in its present order. The authors pass over another one hundred forty pages before taking up the question of the double points of an involution. Consequently, had they chosen the von Staudt definition, there would have been absolutely no way of telling whether a conic had any points or not. The latter part of the chapter goes to Desargues' involution theorem, and to pairs of conics. There is a certain vagueness in the statement of the former which should be remedied. We read (page 129) "*Any line, not through a vertex of the determining quadrangle, is met by the conics of a pencil of Type I in the pairs of an involution.*" Now as this sentence stands, it says emphatically that every line in the plane of a conic will meet that conic at least once, an admission which the authors have no wish to make at present. What they mean is that those conics of the pencil which meet the line do so in pairs of an involution.

An important change comes over the face of the book with Chapter VI, where the algebraic treatment begins. We start with two single-valued operations on pairs of the points of a line called *addition* and *multiplication*. These are shown to be commutative, distributive, and associative. This premised, let us turn to arithmetic. The element here is called a *number*. It is not requisite that a number should be a class of classes, or a recognizable graphical symbol; what is important is that the assemblage of numbers should be subject to two operations $\oplus$ and $\odot$ which follow the five laws mentioned above. The content of the class is quite immaterial, the essential thing is the nature of the laws of association of the members of the class, or of their symbolic representations on paper. Suppose, now, that every point on a line, except a poor scape-goat called P_∞, be represented symbolically by a finite number of pen-strokes. If we take x and y to stand for two such representations we may replace the symbol for the sum of the corresponding points by $x + y$ and that for the product of the points by xy.

This we believe to be the essence of the abstract isomorphism of point range and number field developed in the present chapter. One or two remarks may not be amiss. To begin with it is extremely abstract. The student learns at school that a point is that which has position and no magnitude. When in more advanced work he learns that the meaning of the word point is a variable rather than a constant, and in his projective geometry, as here studied, he finds no one cares exactly what a point may be, he will be rudely shaken. When he is told in the present chapter that the same vagueness hangs about the meaning of the word number, he is lucky if he retain any faith at all in God or man. It is probable therefore, that the student will have to go very slowly over all this ground, until the new and very strange ideas are so thoroughly familiar, that he can restate them in his own words; such at any rate was the case with the reviewer. We wonder, secondly, whether there is not another unproved assumption lurking somewhere in the idea that every point on a line may be represented on paper by some recognizable symbol. But here we see the Burali-Forti paradox lurking in ambush, and hasten on to safer ground.

When the correspondence of points on a line and numbers has been established, the whole algebraic apparatus may be set up in short order. Every point on the line has a coordinate, even P_∞ is welcomed by using homogeneous coordinates. The value of the cross ratio $ABCD$ is defined as x when

$$ABCD \;\overline{\wedge}\; P_\infty P_0 P_1 P_x.$$

A linear transformation of the coordinates will give a projective transformation of the line. These algebraic relations are easily extended to the plane (Chapter VII) and to space. Starting with two fundamental points in the plane, every point thereof not collinear with them has two definite coordinates, dependent on the cross ratios of quartets of lines through the fundamental points; a corresponding, though more elaborate, scaffolding will give three coordinates for points in space. We feel that the handling of this part of the subject is somewhat lacking in grace. Non-homogeneous point coordinates are never subsequently used, and might as well be omitted. The treatment of homogeneous coordinates seems less elegant than that given long ago in Pasch's "Neuere Geometrie."

We are not yet, by any means, through with one-dimensional projectivities. They return in force in Chapter VIII. We learn that projectivities with common fixed points are commutative. Then comes an elaborate discussion of projectivities and involutions on a conic, including the important theorem (page 220) "The group of projective collineations in a plane leaving a non-degenerate conic invariant is simply isomorphic with the general projective group of a line." The whole chapter is full of interesting and valuable material, we only regret that the algebraic aspect of projective transformations is not brought into closer relation with the geometric discussion.

In Chapter IX we reach two new and important ideas, the degree of a geometrical problem and invariance. The authors acknowledge (page 237) considerable indebtedness to Castelnuovo for the treatment of the first of these topics. Whatever the source, the result is admirable.

But now if we are to do any problems except those of the first degree, we must have some circumstances under which we are sure that a line will intersect a conic. The authors take the fatal plunge in proposition K_2, "*If any finite number of involutions are given in a space S satisfying assumptions AEP, there exist a space S', of which S is a sub-space, such that all the given involutions have double points in S'.*"

This is a special case of proposition K_n already discussed. We fear that some students may find the proof of the latter difficult to follow; if they do not, they are more fortunate than the reviewer. We have already dwelt sufficiently upon the general aspects of this theorem, we therefore drop it with the remark that it gives us no light whatever on the question as to whether a conic and line intersect in S or not.

A third and last discussion of two dimensional collineations appears in Chapter X with a complete classification by means of elementary divisors. It is not made perfectly clear why distinct roots of the characteristic equation might not give the same double point in the collineation, although the proof is very easy. Correlations come in besides collineations, and receive an adequate algebraic treatment. There is a careful discussion of two-dimensional polar systems, and the chapter ends with a complete classification of pairs of conics. Let us repeat for the last time our regret that this clear and satisfactory algebraic handling is so far removed from the corresponding geometric one.

The concluding Chapter XI deals with systems of lines. Here we are frankly disappointed, except with the first and last sixths of the work. After a good discussion of the regulus we have (page 304):

"The lines joining corresponding points on two projective conics in different planes form a regulus, provided the conics determine the same involution I of conjugate points on the line of intersection l of the two planes, and provided the collineation between the two planes determined by the correspondence of the conics transforms l into itself by a projectivity to which I belongs."

The various "provideds" here mean that the two conics have two common points in the great space S', and these correspond to themselves in the projectivity. But like many a man who does not care to reflect on his future state, the authors prefer "One world at a time" and only use S' when they are driven to. Is it worth while to give at all a theorem whose statement is so long, and whose proof, with two corollaries covers six pages of text? The remainder of the geometrical part of the chapter hangs on the following definition (page 311):

"If two lines are coplanar, the lines of the flat pencil connecting them both are said to be *linearly dependent* on them. If two lines are skew, the only lines linearly dependent on them are themselves. On three skew lines are *linearly dependent* all the lines of the regulus of which they are rulers. If l_1, l_2, $\cdots$, l_n are any number of lines and m_1, $\cdots$, m_k are such lines that m_1 is linearly dependent on two or three of the lines l_1, l_2, $\cdots$, l_n, m_2 is linearly dependent on two or three of the lines l_1, l_2, $\cdots$, l_n, m_1, and so on, m_k being linearly dependent on two or three of the lines l_1, l_2, $\cdots$, l_n, m_1, m_2, $\cdots$, m_{k-1}, then m_k is said to be *linearly dependent* on the lines l_1, l_2, $\cdots$, l_n. A set of n lines no one of which is linearly dependent on the others are said to be *linearly independent*."

What the authors here mean is that lines shall be defined as linearly dependent when there is linear dependence among their Plücker coordinates. The reader, however, in facing this definition, is in the position of Pascal, who, in one of his inimitable "Lettres à un Provincial," thus speaks of a theological term, used by his adversaries: "Je fixai ce môt dans ma mémoire, car ça ne répésenta rien à mon intelligence." The whole treatment of linear complexes and congruences is built on this basis (instead of on the null system which enters so naturally under correlations).

We have criticised this book freely; in our opinion no teacher should use it who is not sure enough of his ground to be willing to disagree with the authors on many points. But let no one criticise who is not able to recognise therein a fine piece of American scholarship.

JULIAN LOWELL COOLIDGE.