a number of spheres of radius r so that their centers shall be on the rod and be suitably distributed on it; (4) finally to join by a rod the center of a sphere of radius r inside the frame with the center of a sphere of radius $> r$ (it might as well be $< r$) and by a rod the center of the latter sphere with a sphere of radius r outside the frame, and, then, to locate a number of spheres so that the centers of some of them shall be on one of the rods and those of the rest on the other, the spheres on the former rod suitably increasing in size as their centers approach that of the sphere common to the two rods, whilst the spheres on the second rod suitably decrease in size as their centers recede from the common point of the rods.

The model obviously enables one to distinguish by sight or by touch a sphere range representing a line of S_4 that joins a point inside of an ordinary sphere to an outside point without penetrating the surface from a sphere range representing the case where penetration occurs.

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A DIRECT PROOF OF THE THEOREM THAT THE NUMBER OF TERMS IN THE EXPANSION OF AN INFINITE DETERMINANT IS OF THE SAME POTENCY AS THE CONTINUUM.

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THE following rather obvious theorem is thought of sufficient interest to warrant the publication of a direct proof.

THEOREM: *The number of terms in an infinite determinant is of the potency c of the continuum.*

Proof. It is well known that every infinite determinant can be brought into the form

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & \cdots \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} & \cdots \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdots \end{vmatrix} \quad \text{or } |a_{ij}|.$$

Let the elements of the determinant be ordered according to the sum of their subscripts. In case this sum is the same for two elements these are ordered according to the value of i. Then we get an infinite sequence of elements $a_{11}, a_{12}, a_{21}, a_{13}, a_{22}, \ldots$. In each infinite product of the expanded determinant order the elements which form its factors according to the same principle.

Divide a segment AB into two equal segments AC and CB, CB in two equal segments CD and DB, and so on by sequential halving of the last segment obtaining an infinite sequence of segments whose end points approach B as a limit. Repeat this construction on each of the segments $AC, CD, \ldots$, also on each of the segments thus obtained and so on indefinitely. Consider now a term $a_{i_1j_1}a_{i_2j_2}a_{i_3j_3}\ldots$ of the expanded determinant. Let $a_{i_1j_1}$ be the k_1th element of the determinant when these are ordered as above. Now there is a perfectly definite number of elements of the determinant between $a_{i_1j_1}$ and $a_{i_2j_2}$ which may occur with $a_{i_1j_1}$ as factors in a term of the expanded determinant, that is, which are not in the same row or column with $a_{i_1j_1}$. Let this number be $k_2 - 1$. There is further a definite number of elements between $a_{i_2j_2}$ and $a_{i_3j_3}$ which may occur in a product with $a_{i_1j_1}$, $a_{i_2j_2}$. Let this number be $k_3 - 1$ and so on.

We now assign the term $a_{i_1j_1}a_{i_2j_2}a_{i_3j_3}\ldots$ to a sequence of segments on AB in the following manner: The first segment of the sequence is the k_1th segment of the sequence $AC, CD, \ldots$. The second segment is the k_2th segment of the division of the segment just selected, the third is the k_3th segment of the division of the segment last selected, and so on indefinitely. We thus obtain a sequence of segments corresponding to the term $a_{i_1j_1}a_{i_2j_2}a_{i_3j_3}\ldots$, such that every segment of the sequence lies on all that precede it and also such that the lengths of the segments become indefinitely small as we proceed. Hence this sequence determines a definite point which is a common end point of all the segments of the sequence beyond a certain one or else lies within every segment of the sequence. Note that if it is a common end point of such segments then it is a left end point. In this manner we obtain on AB a point corresponding to every term of the expanded determinant. It remains to show: (1) that we thus obtain a term corresponding to every point on AB, (2) that two points thus set in correspondence to different terms of the expanded determinant are not identical.

(1) Let Q be any point on AB. In the sequence of segments

AC, CD, . . . , there is one segment within which Q lies or of which it is the left end point. (We conceive of the segment AB as extending from left to right.) Further let Q be the left end point of or lie within the k_2'th segment of the division of the k_1'th segment of AB, and so on. Thus we obtain a sequence of numbers k_1', k_2', k_3', . . . , corresponding to a definite sequence of segments which determine the point Q. The term $a_{i_1'j_1'}$ $a_{i_2'j_2'}a_{i_3'j_3'}$ . . . is set in correspondence with the point Q by the process described above, provided $a_{i_1'j_1'}$ is the k_1'th element of the determinant, $a_{i_2'j_2'}$ the k_2'th element after $a_{i_1'j_1'}$ of those which can occur with it in a term of the expanded determinant, and so on.

(2) Two different terms $a_{i_1j_1}a_{i_2j_2}a_{i_3j_3}$ . . . and $a_{i_1'j_1'}a_{i_2'j_2'}$ $a_{i_3'j_3'}$ . . . of the expanded determinant cannot be set in correspondence to the same point by this process. Suppose their kth factors are different. For the sake of simplicity of statement we suppose the first factors $a_{i_1j_1}$ and $a_{i_1'j_1'}$ are different. Then the corresponding points P and P' are end points of or lie within different segments of the sequence AC, CD, . . . , obtained by the division of AB. Hence these points can be identical only in case one, as P, is a right end point of one of these segments and P' the left end point of the next segment. But we noted above that in case a point determined by one of these sequences is a common end point of the segments of the sequence determining it, then it must be a left end point of such segments. Hence P and P' cannot be identical.

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SHORTER NOTICES.

The Theory of Sets of Points. By W. H. YOUNG and GRACE CHISHOLM YOUNG. Cambridge, University Press, 1906. xii + 316 pp.

THIS volume consists of a systematic presentation of the theory of sets of points. The titles of the chapters are: Rational and irrational numbers, Representation of numbers on the straight line, The descriptive theory of linear sets of points, Potency and the general idea of a cardinal number, Content, Order, Cantor's numbers, Preliminary notions of plane sets,