The cases in which the Jacobian determinant of T has at least one non-zero element, say $f_u(0, 0) \neq 0$, are completely discussed. Certain cases where all f_u, f_v, ϕ_u, ϕ_v are zero when $u = 0$, $v = 0$ are treated. If f and ϕ admit a common factor in R, then there is an explosive point in $\bar{R}$, having an infinitely many valued inverse. Even then $\bar{R}$ may be the complete neighborhood of this point, the number of branches which are continuous outside this point being different in different subregions of $\bar{R}$.

53. It is well known that the group of isomorphisms of a group of order p is of order $p - 1$, and that of a cyclic group of order p^2 is of order $p(p - 1)$. The corresponding group of the non-cyclic group of order p^2 is simply isomorphic with the linear homogeneous group on p^2 variables.

The groups of isomorphisms of all types of groups of order p^3 are determined by Western in his paper on "Groups of order p^3q," *Proceedings of the London Mathematical Society*, volume 30.

Professor Marriott has determined the groups of isomorphisms of all types of groups of order p^4. He exhibits these as substitution groups and determines the order of each.

F. N. COLE,
Secretary.

ON THE NEGATIVE DISCRIMINANTS FOR WHICH THERE IS A SINGLE CLASS OF POSITIVE PRIMITIVE BINARY QUADRATIC FORMS.

BY PROFESSOR L. E. DICKSON.

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FOR such a discriminant $-P$, the problem of the representation of numbers by a binary quadratic form of discriminant $-P$ is quite elementary; moreover, factorization into primes is unique in a quadratic field of discriminant $-P$. The only*

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* E. Landau, *Mathematische Annalen*, vol. 56 (1903), p. 671. His method is not applicable to discriminants $-P$, where P is odd, as was pointed out by M. Lerch, ibid., vol. 57 (1903), p. 568. Results obtained by the latter by use of a relation between numbers of classes will here be proved by more elementary means and extensions given.

such discriminants of the form $-4k$ are those having $k = 1, 2, 3, 4, 7$, as was conjectured by Gauss* after an examination of the *determinants* as far as -3000. The present note gives practical criteria and the result of an examination of the values of P less than one and one half million. We denote $ax^2 + bxy + cy^2$ by (a, b, c) and call $b^2 - 4ac$ its discriminant.

First, let $P \equiv 0 \pmod 4$. Then $(1, 0, P/4)$ must be the only reduced primitive form of discriminant $-P$. The case in which $P/4$ is divisible by two distinct primes is excluded, since we may then express $P/4$ as the product of two relatively prime factors a, c, such that $1 < a < c$, and hence obtain the new primitive reduced form $(a, 0, c)$ of discriminant $-P$. Hence $P = 4p^e$, where p is a prime. For $p = 2$, $(4, 4, 2^{e-2} + 1)$ is a primitive reduced form of discriminant $-P$ if $e \geqq 4$, and $(3, 2, 3)$ is one if $e = 3$; while for $e = 1$ or 2, whence $P = 8$ or 16, there is a single primitive reduced form. Next, let $p > 2$. The even number $p^e + 1$ cannot have an odd factor > 1, since otherwise it would equal the product of two relatively prime integers a and c, such that $1 < a < c$, and $(a, 2, c)$ would give a new primitive reduced form of discriminant $-P$. Hence $p^e + 1 = 2^k$. Then $(8, 6, 2^{k-3} + 1)$ or $(5, 4, 7)$ is a primitive reduced form of discriminant $-P$ if $k > 5$ or $k = 5$, respectively. For $k = 4$, $2^k - 1 = 15$ is not a power of a prime. For $k = 1, 2, 3$, $P = 4, 12, 28$, there is a single primitive reduced form.

Next, let $P \equiv 3 \pmod 4$. Then $[1, 1, \frac{1}{4}(1 + P)]$ must be the only reduced primitive form of discriminant $-P$. If $P = rs$, where r and s are relatively prime and > 1, one of the factors is $\equiv 3 \pmod 4$ and the other $\equiv 1 \pmod 4$. Let $r > s$. Then $[(r + s)/4, (r - s)/2, (r + s)/4]$ is a new primitive form of discriminant $-P$, which is reduced if $3s \geqq r$. Its second right neighboring form (obtained by using $\delta = -1$, $\delta' = 0$) is $[s, -s, \frac{1}{4}(r + s)]$, which is reduced if $3s < r$. Hence $P = p^e$, where p is a prime $\equiv 3 \pmod 4$ and e is odd. If $p > 3$, $e \geqq 3$, the form with $a = \frac{1}{4}(p + 1)$, $b = 1$, $c = (p^e + 1)/(p + 1)$ is a new primitive reduced form of discriminant $-P$; indeed, $c > 4a$ since $p^{e-1} \geqq p^2 > p + 2$. For $P = 27$, $(1, 1, 7)$ is the only primitive reduced form. For $P = 3^e$, $[9, 3, \frac{1}{4}(3^{e-2} + 1)]$ or $(7, 3, 9)$ is a primitive reduced form if $e > 5$ or $e = 5$, respectively. Thus, if $P \neq 27$, P must be a prime. Set

$$T_j = \tfrac{1}{4}[(2j + 1)^2 + P] = T_0 + j(j + 1).$$

* Disquisitiones Arithmeticae, Art. 303.

If $j = qm + r$, $0 \leqq r < m$, then $T_j \equiv T_r \equiv T_{m-r-1} \pmod{m}$. For $r > \frac{1}{2}(m-1)$, $m - r - 1 < \frac{1}{2}(m-1)$. Hence any T_j is congruent modulo m to some T_r, where $0 \leqq r \leqq \frac{1}{2}(m-1)$. Let $2g + 1$ be the greatest odd integer $\leqq \sqrt{P/3}$. In a reduced form (a, b, c), $b > 0$, we have $b = 2\beta + 1 \leqq 2g + 1$, $\beta \leqq g$. We shall prove that *there is a single reduced form of discriminant* $-P$ *if and only if* $T_0, T_1, \cdots, T_g$ *are all prime numbers.* If they are primes, a reduced form has $a = 1$, $b = 1$. Conversely, let there be a single reduced form. If T_0 were composite, there would be a reduced form with $b = 1$, $a > 1$. Suppose that $T_0, \cdots, T_{\beta-1}$ are primes, but $T_\beta = ac$, $c \geqq a > 1$, where $0 < \beta \leqq g$. If $a \geqq b$, where $b = 2\beta + 1$, (a, b, c) would be reduced. Hence $a < b$. Applying the above result for $m = a$, we see that $T_\beta \equiv T_r \pmod{a}$, where r is some integer $0 \leqq r \leqq \frac{1}{2}(a-1)$. Thus $r < \beta$, so that T_r is a prime. But $T_r \equiv T_\beta \equiv 0 \pmod{a}$. Hence $T_r = a$. Thus $a \geqq T_0 \geqq \frac{1}{4}(1 + P)$. $P \geqq 3(2g+1)^2 \geqq 3(2\beta+1)^2 > 3a^2$, $a > \frac{1}{4}(1 + 3a^2)$. Thus $(3a-1)(a-1) < 0$, which contradicts $a > 1$.

If P is a prime < 27, then $g = 0$ and the condition is that $T_0 = \frac{1}{4}(1 + P)$ shall be a prime. This is satisfied when $P = 3, 7, 11, 19$.

For $P \equiv 7 \pmod{8}$, $P > 7$, T_0 is even and > 2.

For $P \equiv 3 \pmod{8}$, set $P = 8k - 5$. For $k \equiv 2 \pmod{3}$, $k \geqq 5$, $T_0 = 2k - 1$ is divisible by 3 and exceeds 3; while for $k = 2$, $P = 11$. For $k \equiv 1 \pmod{3}$, P is divisible by 3. For $k \equiv 0 \pmod{3}$, $P = 24t - 5$. For $t \equiv 1, 4$, or $0 \pmod{5}$, $T_0 = 6t - 1$, $T_1 = 6t + 1$ or P is divisible by 5 and exceeds 5 except when $t = 1$, $P = 19$. For $t = 2$ or 3, $P = 43$ or 67 and $g = 1$, while T_0 and T_1 are primes. For $t = 7$, $P = 163$, $g = 3$, and T_0, T_1, T_2, T_3 are primes 41, 43, 47, 53. For $t = 8$, $P = 11 \cdot 17$. There remain the cases $t = 5l + 12$, $5l + 13$, where $l \geqq 0$. Hence we may state the

THEOREM. *There is a single class of positive primitive quadratic forms of negative discriminant* $-P$ *when*

$$P = 3, 4, 7, 8, 11, 12, 16, 19, 27, 28, 43, 67, 163;$$

but more than one class if P *is not one of these* 13 *numbers and not a prime of the form* $120l + 283$ *or* $120l + 307$, $l \geqq 0$.

The remaining primes < 1000 are $P = 283, 523, 643, 883, 307, 547, 787, 907$. For these $g \geqq 4$, $T_2 = 77$, $T_1 = 7 \cdot 19$, $T_0 = 7 \cdot 23$, $T_0 = 13 \cdot 17$, $T_0 = 77$, $T_2 = 11 \cdot 13$, $T_2 = 7 \cdot 29$,

$T_4 = 13 \cdot 19$, respectively. Hence in each case there is more than one class.

A practical method of examining a wide range of values of P consists in first excluding the values of l for which any one of the numbers $T_0, \cdots, T_g$ has a given small prime factor p. For $P = 120l + 283$ or 307, $T_0 = 30l + 71$ or $30l + 77$, $g \geqq 4$. This exclusion has already been effected for $p = 3$ or 5. For $p = 7$, any T_j is congruent to T_0, T_1, T_2 or T_3. For $T_0 = 30l + 77$, these are divisible by 7 if $l \equiv 0, 6, 4, 1 \pmod 7$, respectively; for $30l + 71$, if $l \equiv 3, 2, 0, 4 \pmod 7$. Hence there remain the cases

$$T_0 = 210m + \mu,\ \mu = 137,\ 167,\ 227,\ 101,\ 221,\ 251.$$

The least P is now 403, whence $d \geqq 5$. Now $T_0 \equiv m + \mu \pmod{11}$. Thus T_0 is divisible by 11 if $m \equiv 6, 9, 4, 9, 10, 2 \pmod{11}$, respectively. But $T_k = T_{k-1} + 2k$. Hence if we subtract $2k$ from the m for which $T_{k-1} \equiv 0 \pmod{11}$, we obtain the m for which $T_k \equiv 0 \pmod{11}$. This may be done by counting spaces on square ruled paper. At each point so obtained a hole is punched, thus giving a 6×11 stencil for $p = 11$. The least T_0 is now 221, whence $P \geqq 883$, $g \geqq 8$. Similarly, stencils were constructed for $p = 13, 17, 19, 23, 29$. After using the first three stencils, it was noted that $m \geqq 4$ for each μ, whence $T_0 \geqq 941$, $P \geqq 3763$, $g \geqq 17$.

The first 10710 values of T_0 were examined; to this end m was given the values $\leqq 1785$. The use of each stencil excluded more than half of the values left at the earlier stage. After using the stencils for $p \leqq 29$, we had left 110 numbers, for each of which $T_0, \cdots$, or T_6 was verified to be composite. In just four cases were $T_0, \cdots, T_5$ all prime. The work, including the making of the stencils, was done in two days.

THEOREM. *For* $163 < P < 1{,}500{,}000$ *there is more than one class of positive primitive quadratic forms of discriminant* $-P$.

For a greater P, $g \geqq 353$ and there is more than one class unless $T_0, T_1, \cdots, T_{353}$ are all primes. The chance that such a case will arise is extremely small. Note that, for P not exceeding $1\frac{1}{2}$ millions, $T_0, \cdots, T_{14}$ were shown to be not all prime.