THE CHICAGO MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

At the invitation of the Chicago Section the April meeting of the Society was held this year at the University of Chicago, on Friday and Saturday, the 28th and 29th of the month. This was in many ways a remarkable occasion. It was the first regular meeting of the Society except the summer meetings, to be held away from New York City since its founding there in 1888. And it was arranged that this reunion of the eastern and western members should be especially marked by the delivery of President Bôcher's retiring address, which was postponed for that purpose from the annual meeting of the Society in the preceding December. As was under these circumstances to be expected, the meeting was in every way a most successful and inspiring one. Fifty-three papers filled to overflowing the programme of the four sessions. The attendance numbered 115 persons, including the following 88 members of the Society:

Professor G. N. Armstrong, Professor H. Bateman, Professor W. H. Bates, Professor G. N. Bauer, Professor C. H. Beckett, Professor W. W. Beman, Professor G. A. Bliss, Professor Maxime Bôcher, Dr. R. L. Börger, Professor J. W. Bradshaw, Professor W. C. Brenke, Mr. Daniel Buchanan, Dr. H. E. Buchanan, Dr. Thomas Buck, Dr. H. T. Burgess, Professor W. H. Bussey, Professor D. F. Campbell, Professor H. E. Cobb, Professor F. N. Cole, Dr. A. R. Crathorne, Professor D. R. Curtiss, Professor E. W. Davis, Professor S. C. Davisson, Professor L. E. Dickson, Mr. L. L. Dines, Professor L. W. Dowling, Dr. Arnold Dresden, Professor L. P. Eisenhart, Professor Arnold Emch, Professor Peter Field, Professor H. B. Fine, Professor W. B. Ford, Professor J. W. Glover, Professor A. G. Hall, Professor F. T. H'Doubler, Professor T. F. Holgate, Professor O. D. Kellogg, Professor A. M. Kenyon, Professor Kurt Laves, Dr. N. J. Lennes, Professor A. C. Lunn, Dr. E. B. Lytle, Professor Malcolm McNeill, Dr. W. D. MacMillan, Professor Max Mason, Mr. R. M. Mathews, Professor G. A. Miller, Mr. W. J. Montgomery, Professor C. N. Moore, Professor E. H. Moore, Dr. R. L. Moore, Professor J. C. Morehead, Professor F. R. Moulton, Professor G. W.

Myers, Dr. L. I. Neikirk, Professor W. F. Osgood, Professor G. P. Paine, Professor Alexander Pell, Dr. Anna J. Pell, Professor W. S. Pemberton, Professor A. D. Pitcher, Professor H. L. Rietz, Professor W. J. Risley, Professor W. H. Roever, Mr. R. E. Root, Professor D. A. Rothrock, Professor T. R. Running, Mr. J. M. Rysgaard, Miss I. M. Schottenfels, Mr. A. R. Schweitzer, Professor J. B. Shaw, Professor Mary E. Sinclair, Professor C. H. Sisam, Professor E. B. Skinner, Professor H. E. Slaught, Professor C. S. Slichter, Professor P. F. Smith, Mr. V. M. Spunar, Professor C. E. Stromquist, Professor E. J. Townsend, Professor A. L. Underhill, Professor E. B. Van Vleck, Dr. G. E. Wahlin, Professor C. B. Williams, Professor R. E. Wilson, Professor A. E. Young, Professor J. W. Young, Professor Alexander Ziwet.

President H. B. Fine occupied the chair, being relieved by Professor G. A. Miller and Vice-President G. A. Bliss. The Council announced the election of the following persons to membership in the Society: Professor H. Bateman, Bryn Mawr College; Mr. Samuel Beatty, University of Toronto; Professor J. H. Griffith, University of Michigan; Mr. E. J. Moulton, Harvard University; Mr. George Spitzer, Purdue University; Professor C. J. West, Ohio State University. Eleven applications for membership in the Society were received.

Professor L. E. Dickson was reelected member of the Editorial Committee of the *Transactions* for the term of three years. The President was authorized to appoint a committee to arrange for the summer meeting and colloquium of the Society to be held at the University of Wisconsin in 1913. Provision was also made for a committee to prepare a list of nominations of officers for the annual election in December. It was decided to hold the annual meeting this year in New York City.

On Friday evening seventy-three members gathered at the dinner at the Quadrangle Club, where luncheon was also served on both days of the meeting. These and other lesser gatherings afforded very agreeable opportunities for renewing and widening personal acquaintance and comparing scientific and other notes. The meeting brought together members from Massachusetts to Wyoming and from Minnesota to Missouri and contributed greatly to strengthen the bonds of the Society's common interest.

The Presidential Address of Professor Bôcher, "Charles

Sturm's published and unpublished work on differential and algebraic equations," was delivered at the opening of the afternoon session on Friday. The address will appear in an early number of the next volume of the BULLETIN.

The following papers were read at the Chicago meeting:

(1) Mr. DANIEL BUCHANAN: "A class of periodic solutions of the problem of three bodies, two of equal mass, the third moving in a straight line."

(2) Dr. H. E. BUCHANAN: "An expansion of elliptic functions with applications."

(3) Professor D. R. CURTISS: "Relations between the Gramian, the Wronskian, and a third determinant connected with the problem of linear dependence."

(4) Mr. L. L. DINES: "On the representation of resultants of n polynomials in one variable."

(5) Mr. L. L. DINES: "On the solution of three equations for the three variables in terms of others."

(6) Dr. W. D. MACMILLAN: "A reduction of a system of power series to an equivalent system of polynomials."

(7) Dr. W. D. MACMILLAN: "A method for finding the solutions of a set of analytic functions in the neighborhood of a branch point."

(8) Dr. R. L. MOORE: "On the transformation of double integrals."

(9) Professor MAXIME BÔCHER: Presidential Address — "Charles Sturm's published and unpublished work on differential and algebraic equations."

(10) Professor L. P. EISENHART: "A fundamental parametric representation of space curves."

(11) Professor A. E. YOUNG: "On certain orthogonal systems of lines and the problem of determining surfaces referred to them."

(12) Professor ARNOLD EMCH: "The differential equation of curves of normal stresses in a plane field."

(13) Dr. A. B. FRIZELL: "A set of postulates for well ordered types."

(14) Professor C. J. KEYSER: "Sensuous representation of paths that lead from the inside to the outside of an ordinary sphere in point four-space without penetrating the surface of the sphere."

(15) Professor EDWARD KASNER: "The subdivisions of curvilinear angles."

(16) Mr. R. D. CARMICHAEL: "The general theory of linear q-difference equations."

(17) Mr. R. D. CARMICHAEL: "Note on multiply perfect numbers."

(18) Professor G. A. MILLER: "Isomorphisms of a group whose order is a power of a prime."

(19) Professor R. G. D. RICHARDSON: "Theorems of oscillation for two self-adjoint linear differential equations of the second order with two parameters (second paper)."

(20) Professor J. B. SHAW: "Quaternion functions of three parameters."

(21) Dr. J. E. ROWE: "The combinants of two binary cubics and their geometrical interpretation on the rational cubic curve."

(22) Professor U. G. MITCHELL: "Geometry and collineation groups of the finite projective plane $PG(2, 2^2)$."

(23) Dr. G. E. WAHLIN: "The decomposition of rational primes into ideal prime factors in the field $k(\sqrt[p]{m})$."

(24) Professor L. C. KARPINSKI: "An Italian Algebra of the fifteenth century."

(25) Professor C. H. SISAM: "On hyperconical connexes in space of r dimensions."

(26) Mr. R. E. ROOT: "Iterated limits of functions on an abstract range."

(27) Professor E. B. VAN VLECK: "On the generalization of a theorem of Poincaré."

(28) Professor E. B. VAN VLECK: "On the classification of collineations."

(29) Mr. A. R. SCHWEITZER: "On the philosophy of Grassmann's extensive algebra."

(30) Mr. A. R. SCHWEITZER: "On the 'working hypothesis' in the logic of mathematics."

(31) Professor W. B. FORD: "A set of sufficient conditions that a function may have an asymptotic representation in a given region."

(32) Mr. W. J. MONTGOMERY: "The classification of twisted curves of the fifth order."

(33) Professor WILLIAM MARSHALL: "On Hill's differential equation in the theory of perturbations."

(34) Professor H. BATEMAN: "The fundamental equations of the theory of electrons and the infinitesimal transformation of an electromagnetic field into itself."

(35) Dr. N. J. LENNES: "Curves and surfaces in analysis situs."

(36) Dr. N. J. LENNES: "Extension and application of a theorem of Ascoli."

(37) Dr. L. I. NEIKIRK: "Substitution groups of an infinite degree and their related functions."

(38) Professor JAMES MACLAY: "Parabolic curves."

(39) Mr. J. A. NYBERG: "Projective differential geometry of rational cubic curves."

(40) Mr. E. B. STOUFFER: "Invariants of linear differential equations with applications to ruled surfaces in five-dimensional space."

(41) Dr. W. D. MACMILLAN: "A general existence theorem for periodic solutions of differential equations of a certain type."

(42) Dr. A. R. CRATHORNE: "The catenary with variable end points."

(43) Professor F. R. MOULTON: "Periodic orbits of superior planets."

(44) Professor F. R. MOULTON: "On the curves defined by certain differential equations."

(45) Professor F. H. SAFFORD: "An identical transformation of the elliptic element in the Weierstrass form."

(46) Professor W. H. ROEVER: "Southerly deviation of falling bodies (third paper)."

(47) Professor C. N. MOORE: "Convergence factors in double series."

(48) Professor L. E. DICKSON: "On negative discriminants for which there is a single class of positive primitive binary quadratic forms."

(49) Professor L. E. DICKSON: "On Fermat's 'descente infinie.'"

(50) Professor L. E. DICKSON: "On perfect numbers and Bernoullian numbers."

(51) Professor O. E. GLENN: "On expressing a quantic in terms of assigned powers of a given quantic."

(52) Professor G. R. CLEMENTS: "Implicit functions defined in the neighborhood of a point where the Jacobian determinant is zero."

(53) Professor R. W. MARRIOTT: "Determination of the groups of isomorphisms of the groups of order p^4."

Mr. Nyberg and Mr. Stouffer were introduced by Professor

Wilczynski. Professor Marriott's paper was communicated to the Society through Professor Hallett. The papers of Dr. Frizell, Professor Keyser, Professor Kasner, Mr. Carmichael, Professor Richardson, Dr. Rowe, Professor Mitchell, Professor Karpinski, Mr. Schweitzer, Professor Marshall, Professor Maclay, Professor Safford, Professor Roever, Professor Dickson (first paper), Professor Glenn, Professor Clements, and Professor Marriott were read by title.

Professor Curtiss's paper was published in the June BULLETIN. Mr. Root's paper and Professor Dickson's first paper are contained in the present number of the BULLETIN. Professor Dickson's third paper appeared in the May number of the *American Mathematical Monthly*. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. At the Minneapolis meeting of the Chicago Section Dr. W. D. Macmillan in his paper on "An integrable case in the problem of three bodies" discussed the periodic motion of three bodies, two of equal mass moving in a circle about their center of gravity, the third an infinitesimal moving on a straight line drawn through the center of gravity perpendicular to the plane of motion of the other two. The paper by Mr. Buchanan shows that periodic solutions exist when the third body is considered finite. The solutions are constructed as power series in the mass of the third body, the coefficients being power series in the initial projection from the plane of motion of the other two. The paper shows that periodic solutions exist also when the equal bodies move in ellipses and the third is infinitesimal.

2. The differential equation

$$\frac{d^2x}{dt^2} = -a\{(1+\lambda)x - 2\lambda x^3\}$$

defines x as a doubly periodic function of t. In the first section of his paper Dr. Buchanan shows how to obtain x as a power series in λ, the coefficients of the series being simply periodic functions of t. It is then shown how to use this series in the integration of certain types of linear homogeneous differential equations with doubly periodic coefficients. Lamé's equation is used as an example.

The second section is a discussion of the motion of a heavy particle on the surface of an anchor ring, using methods similar to those discussed in the first section.

4. In his first paper Mr. Dines considers conditions for the existence of a common root of n polynomials in one variable. F. Meyer has shown (*Jahresbericht der Deutschen Mathematiker-Vereinigung*, volume 16) that necessary and sufficient conditions can be stated in terms of the vanishing of $n-1$ functions of the coefficients. These functions are obtained as remainders in a process of successive divisions which is a generalization of the euclidean algorithm for the determination of the greatest common divisor of two polynomials. Foethke, in his dissertation (Königsberg, 1907), has shown by an analysis of the Meyer process that under certain restrictions upon the coefficients of the polynomials, necessary and sufficient conditions can be stated in terms of the vanishing of $n-1$ determinants. Mr. Dines, by a different method, more like the Sylvester method for determining the resultant of two polynomials, shows that necessary and sufficient conditions can always be stated in terms of the vanishing of $n-1$ determinants. If a certain restriction be made involving the coefficients of only two of the polynomials, these $n-1$ determinant resultants can be formed according to a very simple law, their elements being coefficients of the polynomials. If this restricting condition is not satisfied, some of the resultants are complicated, the elements being themselves determinants. In certain cases these more complicated determinants can be reduced to simple ones, and it is possible that this is always true.

5. The second paper of Mr. Dines deals with the solutions of the three equations

$$\phi(u, v, w;\, x_1, \cdots, x_p) = 0, \quad \psi(u, v, w;\, x_1, \cdots, x_p) = 0,$$
$$\chi(u, v, w;\, x_1, \cdots, x_p) = 0$$

for u, v, and w in terms of the x's, where ϕ, ψ, and χ are analytic functions of all their variables in a certain region. If the functional determinant $D(\phi, \psi, \chi)/D(u, v, w)$ is different from zero at a point $(u^0, v^0, w^0;\, x_1^0, \cdots, x_p^0)$ which satisfies the given equations, then as is well known, there is a unique set of analytic functions $u(x_1, \cdots, x_p)$, $v(x_1, \cdots, x_p)$, $w(x_1, \cdots, x_p)$

which reduce to u^0, v^0, w^0 for $x_i = x_i^0$, and which satisfy the given equations identically in the x's. If the functional determinant vanishes at a point which satisfies the given equations, for simplicity say the origin $(0, 0, 0; 0, \cdots, 0)$, there are several sets of solutions $u(x_1, \cdots, x_p)$, $v(x_1, \cdots, x_p)$, $w(x_1, \cdots, x_p)$, which converge to zero with the x's. The number and to some extent the character of these solutions are considered. It is found that the number is in general lmn, where l is the degree of the lowest term of ϕ remaining when the x's are put equal to zero, m is the degree of the lowest term in ψ, and n the degree of the lowest term in χ under the same conditions. The solutions for any one of the variables u, v, w can be found as the solutions of a polynomial of degree lmn in this variable, with the coefficient of the highest power unity, and the coefficients of the other terms power series in the x's vanishing with the x's. This is a generalization of the method given by the Weierstrass factor theorem for obtaining the solutions of a single equation with a single dependent variable, and of the method given by Professor Bliss for obtaining the solutions of two equations in two dependent variables. In a certain neighborhood of the origin, excluding certain point sets of lower dimension than the neighborhood along which some of the lmn sets of solutions coincide, the solutions are found to be analytic.

6. At the Minneapolis meeting of the Chicago Section Dr. MacMillan showed that a system of two power series in two dependent variables could be reduced to a system of two polynomials. In the present paper he shows that a system of any number of power series in the same number of dependent variables and of any number of independent variables can be reduced to an equivalent system of polynomials, provided the power series vanish at the origin and provided also there is a certain determinant distinct from zero. The word "equivalent" is to be understood as meaning that the power series set equal to zero admit the same solutions in the neighborhood of the origin as do the polynomials set equal to zero, and conversely. By means of this reduction he shows that the system has κ solutions in the neighborhood of the origin, κ being the product of the orders of the power series.

7. In his second paper Dr. Macmillan shows that if $x_1, \cdots, x_n$ are defined implicitly as functions of parameters $\mu_1, \cdots, \mu_p$

by means of the equations $f_i(x_1, \cdots, x_n; \mu_1, \cdots, \mu_p) = 0$ ($i = 1, \cdots, n$), where the f_i are analytic in $x_1, \cdots, x_n, \mu_1, \cdots \mu_p$, and regular for $x_1 = \cdots = \mu_p = 0$, and if the terms of lowest degree in the expansions of $f_i(x_1, \cdots, x_n; 0, \cdots, 0)$ are of degree d_i, then the system of equations $f_i = 0$ admits $\prod_{i=1}^{n} d_i$ solutions for $x_1, \cdots, x_n$ which vanish with $\mu_1, \cdots, \mu_p$.

If there is but one parameter μ, these solutions are expansible as power series in integral or fractional powers of μ. In the present paper Dr. MacMillan shows how Newton's polygon can be used for finding the forms of the solutions. The first terms in the expansions of the solutions are determined by the solutions of sets of algebraic equations of degree not exceeding d_i. The terms beyond the first can be obtained by the solution of a set of equations $g_i(\xi_1, \cdots, \xi_r; \mu^k) = 0$ ($i = 1, \cdots, n$), whose functional determinant is in general distinct from zero.

If there are more parameters than one, say $\mu_1, \cdots, \mu_p$ the solutions are not, in general, expansible as power series in integral or fractional powers of $\mu_1, \cdots, \mu_p$.

8. At the last meeting of the Chicago Section Dr. Moore reported on a theorem whose hypothesis may, with some modifications, be stated as follows:

G is a limited point set in space of n dimensions. G_0 is a subset of G of measure 0. $G_{1n}, G_{2n}, \cdots, G_{nn}$ are, for each value of the positive integer n, non-overlapping subsets of G whose interior measures are denoted by $e_{1n}, e_{2n}, \cdots, e_{nn}$ respectively. $r_{1n}, r_{2n}, \cdots, r_{nn}, r'_{1n}, r'_{2n}, \cdots, r'_{nn}$ are real numbers whose totality taken for all values of n is a bounded set of values.

For any given point P and given value of n, i_{Pn} is to have meaning only if there exists a positive integer $i (i \leqq n)$ such that G_{in} contains the point P. In case such value of i exists it is to be designated by the symbol i_{Pn}. If P is a point of G, not belonging to G_0, and such that for an infinite sequence of values of n, i_{Pn} exists (as above defined), then if n is confined to this sequence $\lim_{n=\infty} (r'_{i_{Pn}n} - r_{i_{Pn}n}) = 0$.

Finally $\lim_{n=\infty} \sum_{i=1}^{n} r_{in} e_{in}$ exists.

It was shown that if the above hypothesis is satisfied then the following conclusion results:

$$\lim_{n=\infty} \sum_{i=1}^{n} r'_{in} e_{in} \text{ exists and equals } \lim_{n=\infty} \sum_{i=1}^{n} r_{in} e_{in}.$$

Dr. Moore proposes to apply this theorem to the transformation of double integrals.

10. In 1848 Serret proposed the problem of solving the differential equation $ds^2 = dx^2 + dy^2 + dz^2$, where s, x, y, z are functions of a single parameter, and he obtained a general method of solution. Darboux has discussed the same problem on two occasions, but it remained for Montcheuil, in 1905, to obtain a simple explicit solution, however, as a by-product of an entirely different problem. This solution is

$$x = \phi - u\phi' + \psi', \quad iy = \phi - u\phi' - \psi', \quad z = \phi' + u\psi' - \psi,$$
$$s = \phi' - u\psi' + \psi,$$

where ϕ and ψ are arbitrary functions of u and the primes denote differentiation with respect to u. Professor Eisenhart shows that for the general space curve such a parameter u can be obtained in two ways, the exceptions being that for minimal curves and curves in an isotropic plane there is only one such parameter and that both parameters u are constant for straight lines. When the equations are written in this form, the determination of curves congruent in the euclidean sense reduces to the integration of a Schwarzian equation $\{u, s\} = \phi(s)$; consequently this investigation has close contact with recent work by Study. Curves of zero curvature and curves on the sphere of zero radius are readily studied when the above form of equations is used. One has also at once expressions for minimal curves in 4-space analogous to the Weierstrass formulas for minimal curves in 3-space. It is significant that a necessary and sufficient condition that a curve be algebraic is that x, y, z be algebraic functions of u. When it is desirable to deal entirely with real curves, the equations can readily be put in a more convenient form.

11. In a recent paper (*American Journal of Mathematics*, volume 32, No. 1), Professor Young discussed the problem of the spherical representation from the standpoint of the point coordinates, making use of the fundamental equations in

the Bonnet form. The present paper in many respects is a generalization of that. There the lines of reference were assumed to be lines of curvature on the surfaces, a condition which simplified considerably the equations involved and limited the results of solvable cases to particular surfaces. In the present paper, assuming in the first place that the lines of reference are merely orthogonal, a classification of certain such lines is made, based upon the fact that if these are chosen as lines of reference the most important of the fundamental equations, a Laplace equation, is integrable. These lines, which appear upon all surfaces, are called invariant lines as they are defined by equations which require that an invariant of a Laplace equation which belongs to the sequence formed from the given Laplace equation shall vanish.

A second classification of orthogonal lines, principal lines, as they are called, has been made, and the basis for this classification is the fact that the determination of such a set of lines on a surface makes possible the determination of another by algebraic methods from the known functions of the first. The second surface is an associate of the former, and hence its determination makes possible a deformation. Principal lines may or may not be invariant lines. Lastly, Professor Young has discovered a method for determining whether a surface is isothermic or not by algebraic methods requiring no quadrature and no reference to lines of curvature. This makes possible the extension of a very important theorem for the determination of a sequence of isothermic surfaces from any given one, so that the lines of reference need no longer be lines of curvature but may be any orthogonal system of lines whatsoever.

12. The stresses around a point (x_1, y_1) of a plane field are determined if the stresses acting on any two sections through the point are known. Assuming any analytic curve $y-y_1=f(x-x_1)$ referred to cartesian coordinates, the stress P at the point (x_1, y_1) acting on a section tangent to the curve will be an analytic function of the coordinates and the slope

$$P=\phi\left(x_1, y_1, \frac{dy}{dx}\right). \tag{1}$$

Designating the slope of P by dy'/dx' it is known that the directions of corresponding sections and stresses form an involutoric pencil, so that

$$(2) \qquad \frac{dy'}{dx'} = \frac{a \cdot dy/dx + b}{c \cdot dy/dx - a},$$

where a, b, c are functions of x_1 and y_1, determined by the conditions of the field.

The problem of Professor Emch's paper is to determine those curves for which corresponding tangent sections and stresses are normal and which, when considered as perfectly flexible but inelastic under the sole action of these stresses, remain in equilibrium. Writing $dy/dx = p$, the differential equation of this curve is found to be

$$(3) \qquad \frac{dp}{dx}(1 + p^2)^{-\frac{3}{2}} = c \cdot \phi(x, y, p).$$

As an example, the field represented by the hydraulic pressures in a vertical plane through a fluid is worked out. Choosing a cartesian system in this plane with a horizontal and vertical axis, it is found that in this case the curves may be represented by elliptic integrals of the first and second kind. The differential equation for the orthogonal trajectories is obtained by replacing in (3) p in terms of $p' = dy'/dx'$ from (2), i. e., by putting $p' = (ap + b)/(cp - a)$.

13. In Cantor's process of definition the higher well ordered types are inextricably connected with the alephs by the notion of Abbildung. The alephs having proved unmanageable, it seems desirable to reject this idea and proceed simply according to the familiar postulates of arrangement. For example, in the second class, if we write $\alpha' = \omega^\alpha$, $\alpha'' = \omega^{\alpha'}$, etc., no one undertakes to form a working concept of the set of symbols between ω''' and ω^{IV} by counting successive ω-series and assigning to each a limit symbol as starting point for a new one. Instead we insert between $\omega^{\omega''}$ and $\omega^{\omega'''}$ the set ω^α, where α runs through the previously defined set from ω'' to ω''', and then between successive numbers of this new set interpolate in like manner other sets by the rules which yielded the lower symbols. Dr. Frizell makes use of this procedure to obtain a postulational definition of the second and higher classes of ordinal types.

14. By means of a transformation correlating the points of four-dimensional space S_4 with the spheres of ordinary space

S_3, the spheres of real center and pure imaginary radius being included, Professor Keyser's paper shows how to render evident to spatial intuition the existence of point paths in S_4 that lead from the inside to the outside of an ordinary sphere without penetrating the surface and indicates how to make a simple physical model composed of spheres enabling one to distinguish by sight or touch the paths in question from those involving penetration of the surface.

15. Stated in the simplest case, that of bisection, the problem discussed by Professor Kasner is as follows: Given an angle whose sides a and b are analytic arcs, to construct through the vertex an arc c so that (Schwarz) reflexion in c shall interchange the sides. The general problem is fundamental in conformal geometry. For angles incommensurable with respect to π, solutions always exist; but for commensurable angles the problem is possible only when certain invariants vanish.

16. By means of a transformation of the form $z = (m_1x + m_2)/(\mu_1x + \mu_2)$, the system of functional equations

$$(1) \qquad H_i\left(\frac{az+b}{cz+d}\right) = \sum_{j=1}^{n} \lambda_{ij}(z)H_j(z) \qquad (i = 1, \cdots, n)$$

in the n unknown functions $H_1(z), \cdots, H_n(z)$ may be transformed into the system of difference equations

$$(2) \qquad G_i(x+1) = \sum_{j=1}^{n} \bar{\lambda}_{ij}(x)G_j(x) \qquad (i = 1, \cdots, n)$$

or into the system of q-difference equations

$$(3) \qquad G_i(qx) = \sum_{j=1}^{n} \bar{\lambda}_{ij}(x)G_j(x) \qquad (i = 1, \cdots, n)$$

according as the substitution $z' = (az + b)/(cz + d)$ has one or two double points. The essential properties of the solutions of (2) are known. (See the author's thesis and a paper by Professor Birkhoff in the *Transactions*, April, 1911, where references to the earlier literature are given.)

In the present paper, Mr. Carmichael proves the existence and obtains the fundamental properties of the solutions of a system of q-difference equations (3) for the case $|q| \neq 1$. In many respects the theory is analogous to that of system (2), and the two theories interact in such a way that each throws light on the other. There is a fundamental system of solutions of simple character at zero and one of simple character at infinity. A study of the relations between these two systems of solutions gives rise to a theory analogous to the remarkable Birkhoff characterization of the solutions of a system of difference equations.

17. If the sum of all the divisors of an integer N is mN, where m is an integer, N is said to be a multiply perfect number of multiplicity m. Forty-one multiply perfect numbers are known; the most of them have been discovered by Euclid, Mersenne, Fermat, Descartes, Lucas, Lehmer, and Carmichael. It appears that the only rules heretofore known for determining these numbers, when the multiplicity is greater than 2, are those of Descartes (Oeuvres 2, pages 427–429); and even these are effective in only a few cases. After a short historical account of these numbers, Mr. Carmichael gives some simple rules which may be applied to known multiply perfect numbers in order to find others. The effectiveness of the rules in a favorable case is seen in the fact that if they are applied to a certain number given by Lehmer they lead at once to thirteen other multiply perfect numbers, all of which are new. The paper closes with a table of multiply perfect numbers. This credits to the discoverer each of the forty-one numbers which have previously been published. It contains in addition forty-two numbers which are new.

18. After observing that the group of isomorphisms of a group of order p^m cannot involve any operator whose order is divisible by p^m, Professor Miller determines all the groups of order p^m whose groups of isomorphisms involve operators whose orders are either p^{m-1} or p^{m-2}, p being any prime number. When p is odd, the cyclic group of order p^m is the only possible group of this order whose group of isomorphisms involves operators of order p^{m-1}. When $p = 2$, the group of isomorphisms of the cyclic group of order p^m involves no operator whose order exceeds p^{m-2}, but both the dihedral and the dicyclic group of order p^m have operators of order p^{m-1} in their groups of isomorphisms. Hence there are exactly two groups of order 2^m, $m > 3$, which have operators of order 2^{m-1} in their groups of isomorphisms.

A necessary and sufficient condition that the group of isomorphisms of a group of order p^m, $p > 2$ and $m > 4$, involves operators of order p^{m-2} is that the group itself involves operators of order p^{m-1}. Hence there are two and only two groups of order p^m, $p > 2$ and $m > 4$, which have operators of order p^{m-2} in their groups of isomorphisms without having also operators of order p^{m-1} in these groups. When $p = 2$ and $m > 5$, there are 13 groups of order p^m whose groups of isomorphisms involve operators of order p^{m-2}. An important theorem which was proved incidentally may be stated as follows:

If a group of order p^m, $p > 2$ and $m > 4$, contains no operator of order p^{m-1}, the number of its subgroups of order p^4 which do not include any operator of order p^3 is of the form $1 + kp$.

In particular, if a group of order p^m, $p > 2$, and $m > 4$, does not contain any operator of order p^{m-1}, it contains an invariant subgroup of order p^4 which does not involve any operator of order p^3.

19. As an extension of the results reported to the Society in December 1910, Professor Richardson has derived necessary and sufficient conditions for the existence of solutions $u(x)$, $v(y)$, of the equations

$$(p_1u')' + q_1u + (\lambda A_{11} + \mu A_{12})u = 0,$$

$$(p_2v')' + q_2v + (\lambda A_{21} + \mu A_{22})v = 0 \qquad (p_1(x) > 0,\ p_2(y) > 0),$$

which satisfy the boundary conditions $u(a_1) = u(b_1) = 0$, $v(a_2) = v(b_2) = 0$ and oscillate m and n times respectively. The problem is divided into six cases corresponding to different forms of the functions A_{11}, A_{12}, A_{21}, A_{22}. In two of these cases there always exist solutions; in the other four the necessary and sufficient conditions are nearly identical. For example, when $A_{12} \geqq 0$, $A_{22} \geqq 0$, A_{11}, A_{21} arbitrary, it is necessary that

$$\min \frac{A_{11}}{A_{12}} \geqq \min \frac{A_{21}}{A_{22}}, \quad \max \frac{A_{11}}{A_{12}} \geqq \max \frac{A_{21}}{A_{22}}$$

or that

$$\min \frac{A_{11}}{A_{12}} \leqq \min \frac{A_{21}}{A_{22}}, \quad \max \frac{A_{11}}{A_{12}} \leqq \max \frac{A_{21}}{A_{22}}.$$

When the equality sign is omitted these conditions are sufficient.

The oscillation theorem for the equation

$$(py')' + qy + \lambda Ay = 0 \qquad (p(x) > 0,\ y(0) = y(1) = 0),$$

where $q(x)$ is positive in at least a part of the interval and $A(x)$ takes both signs, may be stated as follows: The necessary and sufficient condition that there exist a solution which oscillates n times in the interval is that $n \geqq n_1$, where n_1 is the number of zeros within the interval 0, 1 of that solution of $(py')' + qy = 0$ which vanishes at $x = 0$.

20. In Professor Shaw's paper the quaternion q is considered as a function of three scalar parameters u_1, u_2, u_3,

$$q = f(u_1, u_2, u_3).$$

The properties of the related operator

$$\Delta = \left(A_3 \cdot nq_2q_3 \cdot \frac{\partial}{\partial u_1} + A_3 \cdot nq_3q_1 \cdot \frac{\partial}{\partial u_2} + A_3 \cdot nq_1q_2 \cdot \frac{\partial}{\partial u_3}\right) / \tau n^2$$

and of

$$\delta = \left(A_2 \cdot q_2q_3 \cdot \frac{\partial}{\partial u_1} + A_2 \cdot q_3q_1 \cdot \frac{\partial}{\partial u_2} + A_3 \cdot q_1q_2 \cdot \frac{\partial}{\partial u_3}\right) / \tau n$$

are considered. The differential parameters for a function $\psi(u_1, u_2, u_3)$, dependent for its value on the same scalars as q, are expressed in terms of δ. The differential geometry of a curvilinear three-dimensional space in a four-dimensional flat space furnishes an application. The paper is a first generalization of that read at Minneapolis in December.

21. In Dr. Rowe's paper the equations of a rational cubic curve R^3 are written

$$(1) \qquad x_i = a_it^3 + 3b_it^2 + 3c_it + d_i, \qquad (i = 0, 1, 2).$$

Two lines $(\xi x) = 0$ and $(\eta x) = 0$ cut (1) in the two binary cubics

$$(2) \qquad (a\xi)t^3 + 3(b\xi)t^2 + 3(c\xi)t + (d\xi) = 0,$$

$$(3) \qquad (a\eta)t^3 + 3(b\eta)t^2 + 3(c\eta)t + (d\eta) = 0,$$

$$[(a\xi) = a_0\xi_0 + a_1\xi_1 + a_2\xi_2].$$

The combinants of (2) and (3) are expressible in terms of the two-row determinants of the matrix

$$(4) \qquad \left\| \begin{matrix} (a\xi) & (b\xi) & (c\xi) & (d\xi) \\ (a\eta) & (b\eta) & (c\eta) & (d\eta) \end{matrix} \right\|,$$

and if the coordinates of the point in which (ξx) and (ηx) intersect are replaced by x_0, x_1, x_2 respectively, then any two-row determinant of (4) such as $\left|\begin{smallmatrix}(a\xi) & (b\xi) \\ (a\eta) & (b\eta)\end{smallmatrix}\right|$ becomes the three-row determinant

$$(5) \qquad | abx | = \begin{vmatrix} a_0 & b_0 & x_0 \\ a_1 & b_1 & x_1 \\ a_2 & b_2 & x_2 \end{vmatrix}.$$

Consequently, corresponding to the combinant of (2) and (3) are curves which are covariants of the R^3.

The Jacobian of (2) and (3) is a binary quartic in t whose coefficients are of degree one in three-row determinants of the type (5). Let us call it $(\alpha t)^4 = 0$. Its roots are the parameters of the four tangents from any point to the R^3.

If we form the osculant conic of (1) at a point t', its point equation contains t' to the 4th power. If now t' is made variable, we have a second binary quartic in t whose coefficients are of degree two in determinants of type (5). Let us call it $(\beta t)^4 = 0$; for any given point this gives the parameters of the four osculant conics through the given point.

If P is the covariant line of R^3 cutting out its flexes, and Q the product of the three flex tangents of the R^3, then the simultaneous invariants of $(\alpha t)^4 = 0$ and $(\beta t)^4 = 0$, which are combinants of (2) and (3), are expressible in terms of P and Q. In this way covariant curves of the R^3 defined by projective relations among the roots of $(\alpha t)^4 = 0$ and $(\beta t)^4 = 0$ are expressed in terms of P and Q and their geometrical interpretation is given.

22. It is the purpose of Professor Mitchell's paper to discuss some of the properties of the finite projective plane $PG(2, 2^n)$ and in particular to determine all subgroups of the group of all projective collineations in $PG(2, 2^2)$. If an outside point of a conic be defined as any point of intersection of tangents to the conic, it is found that in $PG(2, 2^n)$ every conic has but one outside point and all tangents to the conic concur at that point. The types of collineations in $PG(2, 2^n)$ are found to be the same as in the ordinary projective plane, except that in Type I all of the fixed elements may be imaginary. It is shown that every subgroup of the group G_{60480} of all projective collineations in the $PG(2, 2^2)$ except a self-conjugate G_{20160} leaves in-

variant a real figure or an imaginary triangle. Subgroups are determined which leave invariant a point, a line, and an imaginary triangle. The other subgroups are the simple G_{360}, the Hessian G_{216}, the simple G_{168}, and their subgroups. The G_{360}, G_{216}, and G_{168} are all consequently subgroups of the G_{60480} and in the $PG(2, 2^2)$ the geometric invariant of each group is a real configuration.

23. In volume 11, number 4 of the *Transactions*, Professor Westlund published an article on the fundamental number of the field $k(\sqrt[p]{m})$ in which he showed the nature of the decomposition of p and the rational prime factors of m in this field. In the present paper Dr. Wahlin shows how any rational prime, different from these, is decomposed into ideal prime factors in the same field, giving the number of factors and their degrees.

24. The Plimpton library contains several algebraic manuscripts of the fifteenth century. Doubtless the most important of these is the Italian manuscript on vellum which contains numerous biographical notes of Italian teachers of mathematics prominent in the fourteenth and fifteenth centuries. The study made by Professor Karpinski of this Algebra shows that the first chapter dealing with algebra is largely a translation of Mohammed ben Musa Al-Khowarizmi's Algebra. This may well be the translation supposed to have been made by William de Lunis although the reference here to this man is not sufficiently definite to settle the matter. The fourteenth chapter (second on algebra) presents 114 problems from a work on algebra made by an otherwise unknown Maestro Biagio. The fifteenth chapter is from the Algebra of Leonard of Pisa.

This paper will appear in the *Bibliotheca Mathematica*.

25. The name hyperconical connex in space of r dimensions is used in Professor Sisam's paper to denote the locus of the pairs of points (x) and (y) in space of r dimensions whose coordinates satisfy an equation $F(x_0, \cdots, x_r; y_0, \cdots, y_r) = 0$, where F is homogeneous of degree m in $x_0, \cdots, x_r$ and of degree n in $y_0, \cdots, y_r$ and satisfies the condition that, if the point (x) is fixed, the resulting hypersurface $F(y) = 0$ is a hypercone with vertex at (x). It is shown that if $m > n$, $F = 0$ determines the principal coincidence of a point-line connex, and if $m = n$ it determines a line complex. These loci are then studied by means of the above equation.

27. The first paper of Professor Van Vleck gave for a system of linear difference equations a generalization of the following well-known theorem of Poincaré: If the coefficients $P_i(n)$ of the difference equation

$$u(n + m) + P_1(n)u(n + m - 1) + \cdots + P_m(n)u(x) = 0$$

have for $n = + \infty$ the limits A_i, and if the roots of the corresponding auxiliary equation

$$x^m + A_1 x^{m-1} + \cdots + A_m = 0$$

are no two of equal moduli, then the ratio $u(n)/u(n - 1)$ will have a limit for $n = + \infty$, and this limit will be one of the roots of the auxiliary equation.

28. Professor Van Vleck's second paper sought a basis for the classification of collineations in the iteration of the given finite collineation. This basis has the advantage of bringing into close connection the classifications from the standpoints of Segre and of Lie, and is worked through in detail for space of three dimensions. It is expected that the paper will appear in the near future, and a lengthy abstract is therefore omitted.

29. Mr. Schweitzer investigates the philosophy which underlies the introduction of Grassmann's Ausdehnungslehre of 1844. Its most striking feature is perhaps its manifoldness: conceptions of Parmenides, Heracleitos, Plato, Aristotle, Descartes, Kant, Jacobi (cf. Fries) are clearly evident. The key to this variety is probably found in the mediating philosophy of Schleiermacher, who seems to have had fundamentally a Heracleitean motive. Without attempting to express in final or most reduced form the set of general conceptions of Grassmann's philosophy, the author roughly divided these into two classes, objects and acts.

I. Objects: opposition and harmony, being and becoming, representation (copy), flow, change, uniformity, law, valuation, purpose, predominance. Subsidiary to opposition are: within and without, preceding and following, general and particular, one and many, identity and diversity.

II. Acts: thought, division, analysis, generation, synthesis, giving and coupling.

In his well-known treatise on geometry Veronese seems closely to follow Grassmann's philosophical introduction as a general basis for his specific development.

30. In his second paper Mr. Schweitzer considers methods of control of mathematical content. He discussed the necessity of universals for the (genetic) logic of mathematics and referred to Grassmann's "guiding conception," the "intuition" of Klein, and Moore's "heuristic principle" as expressions of this need. Finally, Russell's Principles of Mathematics was examined for evidences of induction.

31. Let $f(x)$ be a function of the complex variable x analytic within and upon the boundary of a certain infinite region T, exception being made of the point $x = \infty$. Under these circumstances, $f(x)$ may or may not be developable asymptotically throughout T in the form

$$(1) \qquad f(x) \sim g_0 + g_1 x^{-1} + g_2 x^{-2} + \cdots.$$

It is the object of Professor Ford's paper to determine a set of sufficient conditions for $f(x)$ under which the existence of the development (1) will be assured. Two theorems are established, corresponding to the cases in which x is complex or confined to a real interval, and application is made to certain studies connected with the existence of asymptotic solutions for linear differential equations.

32. It can be shown that with respect to the irreducible surfaces of lowest order which may contain the curve as partial intersection, irreducible twisted quintic curves can be classified as follows :

$$(1)\ C_5 + R_1 = (S_3, S_2), \quad (2)\ C_5 + R_3 = (S_4, S_2),$$
$$(3)\ C_5 + R_4 = (S_3, S_3),$$

a) R_4, quartic of the second kind,
b) R_4, improper quartic of the second kind,

where C_5 denotes the curve in question, R_λ the residual intersection of order λ, and S_μ, S_ν the surfaces of order μ and ν whose complete intersection is $C_5 + R_\lambda$.

Mr. Montgomery effects the classification of all irreducible twisted quintics by means of the system $(m, n, r, h, H, g, G, x, y, \alpha, \beta, \gamma, L)$, which are characteristics of the system of points, lines, and planes of the developable of which the twisted curve is the cuspidal edge. These characteristics are connected by the six Plücker-Cayley equations which, involving 13 quantities, are not sufficient for the complete solution; and consequently two of the characteristics, γ and L, are independently considered. The curves are also classified with respect to monoids, and a special discussion of the quintic with a triple point is given. There are thus obtained 42 different irreducible twisted quintic curves.

33. The object of Professor Marshall's paper is to present a method of solving Hill's differential equation for large values of the argument. A special case is considered in detail, the form of the function determined, and the recurrence formulas for the coefficients given. Incidentally it appears that the results obtained include as special cases the asymptotic expansions for Bessel's functions, for the elliptic cylinder functions, and for the harmonic functions associated with the parabolic cylinder. The application of the method to the general equation is briefly discussed.

34. It was Lie's belief that the processes which take place in nature can be represented mathematically by a succession of infinitesimal transformations for which the laws of the universe are the invariants. In Mr. Bateman's paper, Lie's idea is applied to an electromagnetic field and transformations which convert the field into itself are obtained on the supposition that three particular integral forms remain invariant in the passage from one view of the universe to another.

35. In his dissertation (Chicago, 1907) Dr. Lennes defined a continuous arc connecting two points A and B as a bounded, closed, connected set of points containing A and B but having no connected proper subset containing A and B. In the present paper it is shown that this definition applies without any change whatever to arcs in space. A simply connected closed continuous surface is defined as a closed, bounded, finitely connected set [S] of points such that

(*a*) No simply connected subset separates the remaining points.

(*b*) Every doubly connected subset separates the remaining points into two connected sets.

It is proved that this set of points may be mapped continuously on a tetrahedron and that it separates a three-space in which it lies into two connected sets. It is further shown that obvious modifications of the above definition give a multiply connected surface.

36. Ascoli has shown that if an infinite set of curves connect two fixed points, a necessary and sufficient condition that every infinite sequence of curves of this set shall approach at least one limiting curve uniformly is that the set of curves is *equally* continuous. In Dr. Lennes's second paper this theorem is extended to surfaces and also to curves whose end points are in two separate closed sets of points and not fixed as in Ascoli's theorem.

The generalized theorem is then used to make proofs of very general theorems on the existence of minimizing curves and surfaces. The existence of the limiting curves and surfaces is treated from the non-metric point of view.

37. Dr. Neikirk's paper is a continuation of that presented by him to the Chicago Section at Minneapolis in December. It contains a more detailed study of the properties of infinite substitutions and the generalization of several important theorems on rational functions.

38. From a point (ξ, η) in the plane n tangents proceed to a parabolic curve of the form $y = a_0 + a_1x + \cdots + a_nx^n$. Let k represent the projection on the x axis of the tangent length from the point of contact to (ξ, η). Professor Maclay's paper determines the elementary symmetric functions of the k's in terms of the point (ξ, η) and the coefficients of the curve. The relations of the symmetric functions and functions of the symmetric functions to movements of the point (ξ, η) and to transformations of the plane are treated.

Two results may be stated. For any point (ξ, η) and for every parabolic curve $\Sigma(1/k) = 0$. The second result is expressed in the equation $k_1k_2 \cdots k_n = L/(n-1)a_n$ in which L is the distance from (ξ, η) to the curve measured on a parallel to the y axis. For the ordinary parabola, i. e., $n = 2$, the elementary symmetric functions reduce to the two just given. Interpreted with reference to the parabola, they state that if the parabola be translated parallel to the direction of its principal diameter, the intercepted tangent lengths between the old and new positions project into equal lengths on a line perpendicular to the principal diameter.

39. In his Projective Differential Geometry of Curves and Ruled Surfaces Wilczynski has given a geometrical interpretation

to the seminvariants, semi-covariants, invariants, and covariants of a linear differential equation of the third order. The fundamental solutions y_1, y_2, y_3 of the differential equation are interpreted as homogeneous coordinates of a point P_y in a plane; as the independent variable changes, P_y describes a curve C_y. The semi-covariants, z and ρ, define points P_z and P_ρ in the plane. Taking P_y, P_z, P_ρ as the vertices of a triangle of reference, the equation of the osculating conic at P_y of the curve C_y assumes a very simple form. The application of these interpretations, the determination of the cartesian equation of the osculating conic, of the Halphen point of P_y, and the locus of a covariant expression P_c, when C_y is a cubical parabola, semi-cubical parabola, cissoid, and folium, are the subject of Mr. Nyberg's paper.

40. Professor Wilczynski has proved that the theory of non-developable ruled surfaces in ordinary space can be put into relation with the theory of a system of two linear homogeneous differential equations of the second order. Mr. Stouffer extends this notion to ruled surfaces in five-dimensional space and shows that the projective properties of every surface of this kind, which does not belong to a certain exceptional class, can be obtained by studying a system of two linear homogeneous differential equations of the third order. The ruled surfaces to which the theory does not apply are those which have three consecutive generators in the same space of four dimensions.

41. Given a system of differential equations

$$(A) \qquad \frac{dx_i}{dt} = f_i(x_1, \cdots, x_n;\ \mu;\ t) \qquad (i = 1, \cdots, n)$$

in which the f_i are analytic in x_j and μ and regular for $x_j = \mu = 0$, and, with respect to t, are continuous and periodic with the period 2π; the Cauchy-Poincaré existence theorem states that the solutions of (A) exist and can be expanded as power series in μ and the initial values of $x_1, \cdots, x_n$, the coefficients of these power series being functions of t.

Dr. MacMillan shows that if

$$(B) \qquad \frac{dx_i}{dt} = \sum_{i=1}^{n} \theta_{ij} x_j$$

are the linear terms of (A), then the solutions of (A) are expansible as power series in μ and the constants of integration of the system (B) and converge if these constants are sufficiently small.

By means of the expansions in this form it is shown that if periodic solutions with the period $2\kappa\pi$ as power series in the parameter μ

$$(C) \qquad x_i = \phi_i(\mu;\, t)$$

can be found which formally satisfy (A), then the solutions (C) exist and converge for all values of t provided μ is sufficiently small.

42. The problem of the position of equilibrium of a heavy cord with one end point fixed and the other movable on a given curve has been discussed by Kneser as a problem in the calculus of variations. Dr. Crathorne considered this problem for both end points movable on given curves K_1 and K_2, and in particular discussed the Jacobi condition. The resulting curve is a catenary intersecting K_1 and K_2 at right angles. The relations between the critical points and the curvature of K_1 and K_2 at their intersections P_1 and P_2 with the catenary were given. In the case in which the catenary approaches the convex side of both the fixed curves the Jacobi condition is satisfied. In the other cases the fulfillment of this condition depends on the curvatures of K_1 and K_2 at P_1 and P_2. Also, if P'_1 and P'_2 represent the critical points of K_1 and K_2 respectively, then P'_1 cannot lie between P_1 and P'_2. This is analogous to Bliss's condition in the non-isoperimetric problem.

43. Periodic solutions of the problem of three bodies fall into a large number of classes which require somewhat different modes of treatment. The class of orbits considered by Professor Moulton in this paper is that in which two finite bodies describe circles, while a distant infinitesimal body moves in the plane of their motion subject to their attraction. The orbits discussed are those whose period is the synodic period of revolution of the infinitesimal body. The solutions are regular analytic functions of various parameters among which is the energy constant. For a given value of the constant there are six geometrically distinct orbits in three of which the motion is direct and in three of which it is retrograde. For small values of the parameter only two of the orbits are real.

44. The differential equation treated in Professor Moulton's second paper is

$$\frac{d^2x}{dt^2} = \sum_{i=-\infty}^{+\infty} a_i x^i,$$

where the a_i are constants. This equation includes as special cases those defining elliptic and hyperelliptic functions, as well as many others which arise in physical problems. Discussion is made particularly of the solutions which are periodic. Convenient methods of obtaining their periods and explicit representations are developed. In case the right member is a polynomial in x the precise domain of validity is found. These results include new and useful expansions of the elliptic functions. The Legendre elliptic functions of real period are explicitly developed to the twelfth power of the modulus inclusive, and it is proved they are valid for all real finite values of t.

45. The transformation employed in Professor Safford's paper is quadric and irreducible. It is based upon a formula given in Weierstrass's lectures and published in a dissertation by G. G. A. Biermann. This formula may also be found in Enneper, Elliptische Functionen, § 5, and in Greenhill, Elliptic Functions, § 150.

46. Professor Roever derived for the southerly deviation of a falling body the expression

$$(1) \qquad \frac{4 \sin \phi \cos \phi \omega^2 - 5(\partial g/\partial \xi)_0}{6g_0} h^2,$$

in which h is the height through which the body falls, ϕ and g_0 the latitude and the acceleration respectively at the point P_0 from which the body falls, ω the angular velocity of the earth's rotation, and ξ (southerly) distance measured along the meridian which passes through P_0. If $f_1(r, z)$ (where r represents the distance of a general point from the earth's axis of rotation and z that from a plane π perpendicular to this axis) represents the potential function of the field of force F_1, under the influence of which the falling body moves after its connection with the rotating earth has been severed,*

* The potential function of the field of force F_2, under the influence of which the plumb-line is in equilibrium, is $f_2(r, z) = f_1(r, z) + \frac{1}{2}\omega^2 r^2$.

$$g_0 = \sqrt{\left(\frac{\partial f_1}{\partial z}\right)_0^2 + \left[\left(\frac{\partial f_1}{\partial r}\right)_0 + \omega^2 r_0\right]^2}, \quad g_0 \sin\phi = \left(\frac{\partial f_1}{\partial z}\right)_0,$$

$$g_0 \cos\phi = \left(\frac{\partial f_1}{\partial r}\right)_0 + \omega^2 r_0,$$

$$\left(\frac{\partial g}{\partial \xi}\right)_0 = \sin\phi \cos\phi \left[\left(\frac{\partial^2 f_1}{\partial z^2}\right)_0 - \left(\frac{\partial^2 f_1}{\partial r^2}\right)_0\right]$$

$$+ (\cos^2\phi - \sin^2\phi)\left(\frac{\partial^2 f_1}{\partial r \partial z}\right)_0 - \sin\phi\cos\phi\,\omega^2,$$

the subscript $_0$ denoting that the derivatives are computed for the point $P_0(r = r_0, z = z_0)$. Expression (1) is equal to the difference $\xi_1 - \xi_2$, where

$$\xi_1 = \frac{4 \sin\phi \cos\phi\,\omega^2 + (\partial g/\partial\xi)_0}{6 g_0} h^2, \quad \xi_2 = \frac{(\partial g/\partial \xi)_0}{g_0} h^2.$$

In order to show what ξ_1 and ξ_2 represent, let us denote by R the plumb-bob of the plumb-line which is supported at P_0 and has the length $h = P_0R$, and by T and S the points in which the direction of gravity at P_0 and the path of the falling body pierce the level surface of R, respectively. Then ξ_1 and ξ_2 are the distances, measured along a meridian, which the circles of latitude of S and R lie to the south of the circle of latitude of T, and the southerly deviation of the falling body is the distance which the circle of latitude of S lies to the south of that of R. In existing formulas the part ξ_2 has been entirely neglected. In the formula of Gauss ($\frac{2}{3}\sin\phi\cos\phi\,\omega^2/g_0 \cdot h^2$) this is due to the assumption that g is constant and therefore that $(\partial g/\partial \xi)_0 = 0$. For the potential function for which the standard spheroid is a level surface,† $-\xi_2$ is eighteen times as great as ξ_1, and the southerly deviation $\xi_1 - \xi_2$ is four and one half times as great as the expression given by Gauss.

47. Cesàro's definition of a series k times indeterminate for the case of a simple series can be extended to double series.

† $f_1(r, z) = \frac{M}{\rho}\left[1 + \frac{\varepsilon}{3}\frac{\rho_1^2}{\rho^2}\frac{r^2 - 2z^2}{\rho^2}\right]$,

where $\rho^2 = r^2 + z^2$, $\varepsilon = .00167$, M and ρ_1 are mass and mean radius of earth. See Poincaré, Figures d'Equilibre d'une Masse fluide (1902), Chap. V.

We form

$$S^{(k)}_{m,n} = \sum_{i=0}^{i=m}\sum_{j=0}^{j=n}\binom{k+m-i-1}{k-1}\binom{k+n-j-1}{k-1}s_{i,j},$$

where

$$s_{m,n} = \sum_{0\ 0}^{m,\ n} a_{i,j},$$

and

$$A^{(k)}_{m,n} = \binom{m+k}{k}\binom{n+k}{k}.$$

Then if $S^{(k)}_{m,n}/A^{(k)}_{m,n}$ approaches a limit as m and n become infinite, whereas $S^{(k-1)}_{m,n}/A^{(k-1)}_{m,n}$ does not, we say that the series $\sum a_{i,j}$ is k times indeterminate, and that its sum is equal to the above limit.

The principal result of Professor Moore's paper is the proof of the following theorem: If the series $\sum a_{i,j}$ is k times indeterminate with sum S and furthermore $S^{(k)}_{m,n}/A^{(k)}_{m,n}$ remains finite for all values of m and n, then the series $\sum a_{i,j}f_{i,j}(\alpha, \beta)$ will converge for positive values of α and β, its value $F(\alpha, \beta)$ will be continuous for such values, and finally

$$\lim_{\alpha=+0,\ \beta=+0} F(\alpha, \beta) = S,$$

provided the convergence factors $f_{i,j}(\alpha, \beta)$ are continuous for $\alpha, \beta > 0$, and satisfy the following conditions:

$$(a)\qquad \sum_{\mu_1,\ \nu_1}^{\mu_2,\ \nu_2} i^k j^k \,|\,\Delta_2^{k+1} f_{i,j}\,| < K, \qquad \alpha, \beta > 0,$$

$$(b)\qquad \lim_{m=\infty} m^k f_{m,n} = 0, \quad \lim_{n=\infty} n^k f_{m,n} = 0, \quad \lim_{\substack{m=\infty\\ n=\infty}} m^k n^k f_{m,n} = 0,$$

$$(c)\qquad \lim_{\alpha=+0,\ \beta=+0} f_{i,j}(\alpha, \beta) = 1 \quad (i = 1, 2, 3, \cdots; j = 1, 2, 3, \cdots),$$

where K is a positive constant, μ_1, ν_1, μ_2, ν_2 are any four positive integers, and $\Delta_2^{k+1} f_{i,j}$ is used as an abbreviation for

$$\sum_{\mu=i}^{\mu=i+k+1}\ \sum_{\nu=j}^{\nu=j+k+1} (-1)^{\mu-i}(-1)^{\mu-j}\binom{k+1}{\mu-i}\binom{k+1}{\nu-j} f_{\mu,\nu}.$$

49. The second paper by Professor Dickson deals with Fermat's sketch of his proof that no right triangle with rational sides can equal a square with a rational side. Fermat (Oeuvres

3, Paris, 1896, page 271) states that if $\xi^2 = \eta^2 + 2b^2$ it is easily proved that ξ itself is of the form $r^2 + 2s^2$. It is of the nature of an anachronism to complete the proof, as did Legendre, Théorie des nombres, Paris, 1808, page 342, by means of the theory of quadratic forms, or, as did Bachmann, Niedere Zahlentheorie 2, page 451, by means of the properties of the integral algebraic numbers $x + y\sqrt{-2}$. Very likely Fermat had a proof of the following nature. Set $e = \frac{1}{2}(\xi + \eta), f = \frac{1}{2}(\xi - \eta)$. Then e, f, and $ef = \frac{1}{2}b^2$ are integers, and e, f are relatively prime. By changing the sign of η, if necessary, we may assume that e is odd. Then $e = r^2, f = 2s^2, \xi = e + f = r^2 + 2s^2$.

Fermat then considers the right triangle with the legs r^2 and $2s^2$ (and hence of area a square). Its legs may be expressed in the form $2m_1n_1 = 2s^2$ and $m_1^2 - n_1^2 = r^2$, where $m_1 = a_1^2, n_1 = b_1^2$, by the earlier argument. Then $(m_1^2 + n_1^2)^2 = r^4 + 4s^4 = a^2$. Hence $a_1^4 + b_1^4 = a$. For integral values, $a_1^2 + b_1^2 \leqq a_1^4 + b_1^4$, $a < a^2 + b^2$. Hence we have proved Fermat's assertion that $a_1^2 + b_1^2 < a^2 + b^2$.

51. The first part of Professor Glenn's paper is devoted to the development of a binary form a_x^n $(n = \mu\nu)$ in a series of ascending powers of a given form α_x^μ. The general development is obtained. The necessary and sufficient conditions are then established that a_x^n be a sum of multiples of prescribed powers of α_x^μ. A special case of these results is Hilbert's covariant which expresses by its identical vanishing the necessary and sufficient conditions that a_x^n be the νth power of α_x^μ (*Mathematische Annalen*, volume 27).

52. Professor Clements considers the transformation

$$T: \qquad x = f(u, v) \quad y = \phi(u, v),$$

f and ϕ denoting functions of the complex variables u and v, single valued and analytic throughout some neighborhood R of $u = 0, v = 0$, and satisfying the conditions

$$f(0, 0) = 0, \quad \phi(0, 0) = 0,$$
$$J(0, 0) = (f_u\phi_v - f_v\phi_u)_{u=0, v=0} = 0 \quad [\not\equiv 0]_{u, v}.$$

The behavior of the inverse solution $u = \bar{f}(x, y), v = \bar{\phi}(x, y)$ in the region $\bar{R}$ for (x, y) into which T carries R, is considered.

It is shown that no inverse can exist, single valued and analytic throughout $\overline{R}$; that if f and ϕ admit no common factor in R, there always exists an inverse, continuous and multiple valued throughout $\overline{R}$, the branches coming together with value zero in $x = 0$, $y = 0$.

A more general set of equations

$$f_i(x_1, \cdots, x_n;\ y_1, \cdots, y_p) = 0 \qquad (i = 1, 2, \cdots, p),$$

where

(1) the f_i are analytic functions of all their arguments in a certain region, and $f_i((0);(0)) = 0$;

$$(2) \qquad \frac{D(f_2, \cdots, f_p)}{D(y_2, \cdots, y_p)} \neq 0 \quad \text{when } (x) = 0,\ (y) = 0;$$

$$(3) \quad J_1[(x);(y)] \equiv \frac{D(f_1, f_2, \cdots, f_p)}{D(y_1, \cdots, y_p)} = 0 \text{ when } (x)=0,\ (y)=0,$$

. . . .

$$J_{k-1}[(x);(y)] \equiv \frac{D(J_{k-2}, f_2, \cdots, f_p)}{D(y_1, \cdots, y_p)} = 0 \text{ when } (x) = 0,\ (y) = 0,$$

$$J_k[(x);(y)] \equiv \frac{D(J_{k-1}, f_2, \cdots, f_p)}{D(y_1, \cdots, y_p)} \neq 0 \text{ when } (x) = 0,\ (y) = 0,$$

defines p functions

$$y_1 = \phi_1(x_1, \cdots, x_n), \cdots, y_p = \phi_p(x_1, \cdots, x_n)$$

of $x_1, \cdots, x_n$, continuous but not analytic throughout some complete neighborhood of $x_1 = 0, \cdots, x_n = 0$. This inverse is multiple valued and has in general k branches. This theorem is applied to the transformation

$$x_i = f_i(y_1, \cdots, y_p) \qquad (i = 1, 2, \cdots, p).$$

Condition (3) requires that the f_i shall have no common factor in the neighborhood of $(y) = 0$. In this case it is shown that the k branches are distinct.

The factorization of transformations is taken up and certain useful types are discussed in detail. It is shown that if T satisfies the further condition $J_2(0, 0) \neq 0$, it can be factored into transformations one-to-one and analytic both ways, plus one of the type $x = u$, $y = v^2$.

The cases in which the Jacobian determinant of T has at least one non-zero element, say $f_u(0, 0) \neq 0$, are completely discussed. Certain cases where all f_u, f_v, ϕ_u, ϕ_v are zero when $u = 0$, $v = 0$ are treated. If f and ϕ admit a common factor in R, then there is an explosive point in $\bar{R}$, having an infinitely many valued inverse. Even then $\bar{R}$ may be the complete neighborhood of this point, the number of branches which are continuous outside this point being different in different subregions of $\bar{R}$.

53. It is well known that the group of isomorphisms of a group of order p is of order $p - 1$, and that of a cyclic group of order p^2 is of order $p(p - 1)$. The corresponding group of the non-cyclic group of order p^2 is simply isomorphic with the linear homogeneous group on p^2 variables.

The groups of isomorphisms of all types of groups of order p^3 are determined by Western in his paper on "Groups of order p^3q," *Proceedings of the London Mathematical Society*, volume 30.

Professor Marriott has determined the groups of isomorphisms of all types of groups of order p^4. He exhibits these as substitution groups and determines the order of each.

F. N. COLE,
Secretary.

ON THE NEGATIVE DISCRIMINANTS FOR WHICH THERE IS A SINGLE CLASS OF POSITIVE PRIMITIVE BINARY QUADRATIC FORMS.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society, April 29, 1911.)

FOR such a discriminant $-P$, the problem of the representation of numbers by a binary quadratic form of discriminant $-P$ is quite elementary; moreover, factorization into primes is unique in a quadratic field of discriminant $-P$. The only*

* E. Landau, *Mathematische Annalen*, vol. 56 (1903), p. 671. His method is not applicable to discriminants $-P$, where P is odd, as was pointed out by M. Lerch, ibid., vol. 57 (1903), p. 568. Results obtained by the latter by use of a relation between numbers of classes will here be proved by more elementary means and extensions given.