number of conditions in order that f_{pm} may be an mth power is $\binom{m+p-1}{m} - p$.

Each one of the Hessians $H\phi_0$, K_j $(j = 1, 2, \cdots, p-2)$ is of order $2m-4$ in the variables which it contains, and so the number of vanishing coefficients in each is $2m-3$. Hence these give $(2m-3)(p-1)$ conditions in addition to the $\binom{m+p-1}{m} - m(p-1) - 1$ assumed ones. But of the $2m-3$ conditions obtained by equating to zero the coefficients of a binary Hessian covariant only $m-1$ are independent, as the m coefficients of the form can all be expressed in terms of a single quantity when the Hessian vanishes. Hence we have as a total number of conditions given by the original factorability conditions of f_{pm} and the Hessians

$$\binom{m+p-1}{m} - m(p-1) - 1 + (m-1)(p-1) = \binom{m+p-1}{m} - p,$$

which is thus the minimum number required. Hence the relations derived in § 2 furnish a minimum set.

In the same way it may be shown that (6), (7), (8), (9) are all minimum sets.

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THE GENERAL TERM OF A RECURRING SERIES.

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1. The principal theorem of this note expresses the general term of a recurring series rationally in terms of the first few terms and the constants of the scale of relation. Although I derived it in 1908, I have only recently learned that practically the same theorem was published by D'Ocagne in 1894 (*Journal de L'Ecole Polytechnique,* volume 64, pages 151–224) and by Netto in 1895 (*Monatshefte für Mathematik und Physik,* volume 6, pages 285–290). Nevertheless it may be worth while to publish my own work for three reasons: first, because my proof is simpler than those of D'Ocagne and Netto; second, because I have stated the result in a more explicit form than that of either of these authors *; third, because I have applied

* D'Ocagne gives an explicit statement of the theorem (p. 163) for the special case in which the series is a "suite fondamentale," but not for the general case.

the result to the series of powers of a matrix, and this application is, I believe, entirely new.*

2. Let $U = u_0 + u_1 + \cdots + u_{n-1} + \cdots + u_m + \cdots$ be any recurring series of order n, and let

$$(1) \qquad u_m = a_1 u_{m-1} + a_2 u_{m-2} + \cdots + a_n u_{m-n} \quad (m = n, n+1, \cdots)$$

be its scale of relation. The general term u_m is evidently a linear homogeneous function of the first n terms $u_0, \cdots, u_{n-1}$ and a rational integral function of the n constants $a_1, \cdots, a_n$ of the scale of relation. Our problem is to determine the explicit form of this function.

The corresponding power series will be

$$(2) \qquad U(x) = u_0 + u_1 x + \cdots + u_{n-1}x^{n-1} + U_n(x),$$

where

$$(3) \qquad U_n(x) = u_n x^n + \cdots + u_{2n-1}x^{2n-1} + \cdots;$$

that is, $U_n(x)$ is obtained from $U(x)$ by removing the first n terms. These series will always be convergent for a certain range of values of x. In all that follows we assume that x is chosen within that range.

From (1) and (3) we easily derive the identity

$$\begin{aligned}(1 - a_1 x - a_2 x^2 - \cdots - a_n x^n) \cdot U_n(x) &= u_n x^n \\ &+ (u_{n+1} - a_1 u_n)x^{n+1} + (u_{n+2} - a_1 u_{n+1} - a_2 u_n)x^{n+2} \\ &+ \cdots + (u_{2n-1} - a_1 u_{2n-2} - \cdots - a_{n-1} u_n)x^{2n-1} \\ &= (a_1 u_{n-1} + \cdots + a_n u_0)x^n + (a_2 u_{n-1} + \cdots + a_n u_1)x^{n+1} \\ &+ \cdots + (a_{n-1} u_{n-1} + a_n u_{n-2})x^{2n-2} + a_n u_{n-1} x^{2n-1},\end{aligned}$$

which can be written in the form

$$(4) \qquad U_n(x) = \frac{u'_0 x^n + u'_1 x^{n+1} + \cdots + u'_{n-2} x^{2n-2} + u'_{n-1} x^{2n-1}}{1 - a_1 x - a_2 x^2 - \cdots - a_n x^n},$$

provided we define the auxiliary quantities $u'_0, \cdots, u'_{n-1}$ by the equations

* For other applications see the papers of D'Ocagne and Netto, especially the former.

$$
(5) \qquad
\begin{aligned}
u'_0 &= a_1 u_{n-1} + \cdots + a_n u_0, \\
u'_1 &= a_2 u_{n-1} + \cdots + a_n u_1, \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
u'_{n-2} &= a_{n-1} u_{n-1} + a_n u_{n-2}, \\
u'_{n-1} &= a_n u_{n-1}.
\end{aligned}
$$

The right-hand member of (4) is the so-called generating function of the series $U_n(x)$. We wish to expand it in ascending powers of x. This is easily accomplished, because of the well-known fact that

$$
\frac{1}{1 - a_1 x - a_2 x^2 - \cdots - a_n x^n}
= \sum_{\alpha_1=0}^{\infty} \cdots \sum_{\alpha_n=0}^{\infty} \frac{(\alpha_1 + \cdots + \alpha_n)!}{\alpha_1 ! \cdots \alpha_n !} a_1^{\alpha_1} \cdots a_n^{\alpha_n} x^{\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n}.
$$

Hence, if we define A_m, for every positive integral and zero value of m, by the equation

$$
(6) \qquad A_m = \sum \frac{(\alpha_1 + \cdots + \alpha_n)!}{\alpha_1 ! \cdots \alpha_n !} a_1^{\alpha_1} \cdots a_n^{\alpha_n},
$$

where the summation extends over all the positive integral and zero values of $\alpha_1, \cdots, \alpha_n$, for which $\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = m$, we see that (4) can be written in the form

$$
U_n(x) = \sum_{m=0}^{\infty} A_m (u'_0 x^{m+n} + \cdots + u'_{n-1} x^{m+2n-1}).
$$

Arranging this with respect to x, we have

$$
(7) \quad U_n(x) = \sum_{m=0}^{\infty} (u'_0 A_m + u'_1 A_{m-1} + \cdots + u'_{n-1} A_{m-n+1}) x^{m+n},
$$

provided we agree that

$$
(8) \qquad A_m = 0 \text{ when } m < 0.
$$

Equating coefficients of x^{m+n} in (3) and (7), we obtain *the required formula*

$$
(9) \qquad u_{m+n} = u'_0 A_m + u'_1 A_{m-1} + \cdots + u'_{n-1} A_{m-n+1}
$$

for the general term u_{m+n} *of the recurring series* U, *where the* n *consecutive coefficients* $A_m, \cdots, A_{m-n+1}$, *defined by* (6) *and* (8), *are rational integral functions of the constants* $a_1, \cdots, a_n$ *of the scale of relation, and where the auxiliary quantities* $u'_0, \cdots, u'_{n-1}$, *defined by* (5), *are linear homogeneous functions of the first* n *terms* $u_0, \cdots, u_{n-1}$ *of the series.*

Application to Matrices.

3. Let us now apply the formula (9) so found to the recurring series that consists of the successive positive integral and zero powers of a linear homogeneous substitution in n variables, or in other words of an n-ary matrix $L = (l_{ij})$, where $l_{ij} (i, j = 1, \cdots, n)$ is the element in the ith row and jth column. Let L^0 be the corresponding unit matrix. We wish to express all the powers of L as linear homogeneous functions of the first n powers $L^0, L, \cdots, L^{n-1}$.* The first $n + 1$ powers of L satisfy the well-known † Hamilton-Cayley equation

$$(10) \qquad L^n = a_1 L^{n-1} + a_2 L^{n-2} + \cdots + a_{n-1} L + a_n L^0,$$

where

$$a_1 = \sum_{i=1}^{n} l_{ii}, \quad - a_2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \begin{vmatrix} l_{ii} & l_{ij} \\ l_{ji} & l_{jj} \end{vmatrix} (i < j), \cdots,$$

and

$$(- 1)^{n-1} a_n = | l_{ij} |,$$

the determinant of L. Multiplying (10) by L^{m-n}, we obtain the scale of relation

$$L^m = a_1 L^{m-1} + \cdots + a_n L^{m-n}.$$

Hence we have only to define a set of auxiliary matrices $L_0, L_1, \cdots, L_{n-1}$ by the equations

$$\begin{aligned} L_0 \;\; &= a_1 L^{n-1} + \cdots + a_n L^0, \\ L_1 \;\; &= a_2 L^{n-1} + \cdots + a_n L^1, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ L_{n-2} &= a_{n-1} L^{n-1} + a_n L^{n-2}, \\ L_{n-1} &= a_n L^{n-1}, \end{aligned}$$

* This problem I stated and partly solved in the BULLETIN, vol. 13 (1907), pp. 337–338.

† Cf. Bôcher, Higher Algebra (1907), p. 296.

and our problem is completely solved by the equation

$$L^{m+n} = A_m L_0 + A_{m-1} L_1 + \cdots + A_{m-n+1} L_{n-1},$$

where the A's are scalar quantities defined, as before, by (6) and (8).

To illustrate this method, we shall calculate the 12th power of the ternary matrix

$$L = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 2 \end{pmatrix}.$$

In this case $a_1 = 2$, $a_2 = -2$, $a_3 = 1$,

$$L_0 = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -3 & 2 \\ 2 & -2 & 1 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & 1 & -2 \\ -2 & 4 & -3 \\ -3 & 4 & -2 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -2 & 2 \\ 2 & -3 & 2 \end{pmatrix},$$

$$L^{12} = A_9 L_0 + A_8 L_1 + A_7 L_2,$$

$$A_7 = (a_1^7 + 6a_1^5 a_2 + 10 a_1^3 a_2^2 + 4a_1 a_2^3)$$
$$+ (5a_1^4 + 12 a_1^2 a_2 + 3a_2^2) + 3a_1 a_3^2 = 2;$$

similarly $A_8 = 2$, $A_9 = 1$. Therefore

$$L^{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

CORNELL UNIVERSITY,
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