the designation of the method? Why not call it the "Ruffini-Horner method"?

It should be stated here that Budan, in his famous pamphlet of 1807,* explained a process of transformation of an equation into another whose roots are diminished by h, which bears some resemblance to that of Ruffini and Horner. When the roots are to be diminished by unity, the computation is identical. But when a root is to be diminished by, say 8, there is a difference: Budan does this by eight steps, the roots being diminished by unity in each step. Regarding Budan, Ruffini once wrote to Delambre, the secretary of the French Institute, as follows: †

"Troppo giuste sono le lodi che Ella dà al metodo del Sig. Budan di sciogliere le Equ. numeriche; avrei però desiderato che avesse Ella avuta occasione di vedere la Memoria che sopra lo stesso argomento presentai già alla nostra Società Italiana, che da essa riportò graziosamente il premio e che fu poi stampata nel 1804 fuori degli Atti. Coincido col sig. Budan nella maniera di fare le trasformazioni successive e nel servirmi dei decimali."

THE NEW HAVEN COLLOQUIUM LECTURES.

The New Haven Colloquium. By ELIAKIM HASTINGS MOORE, ERNEST JULIUS WILCZYNSKI, MAX MASON. Yale University Press, 1910. x + 222 pp.

THE fifth Colloquium of the American Mathematical Society was held at New Haven, September 5–8, 1906, under the auspices of Yale University. All the lectures related to fields in which recent progress has been considerable, and were given by men who have made important contributions; on this account the volume which contains the lectures should be of substantial interest, particularly to the American mathematician. Professor Moore gives a first systematic account of what he has termed "a form of General Analysis." Professor Wilczynski outlines the point of view and some of the principal results in

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* F. D. Budan, Nouvelle méthode pour la résolution des équations numériques, Paris, 1807, pp. 14, 15, 29, 39.

† *Memorie della Società Italiana delle Scienze* (detta dei XL), Serie 3ª, Tomo XIV, 1906, p. 296.

projective differential geometry. Professor Mason treats a variety of boundary value problems.

I. *Introduction to a Form of General Analysis.*

It is obvious to those who have been following recent mathematical progress that, since the researches of Hill, Volterra, and Fredholm in the direction of extended linear systems of equations, mathematics has been in the way of a great development. That attitude of mind which conceives of the function as a generalized point, of the method of successive approximation as a Taylor's expansion in a function variable, of the calculus of variations as a limiting form of the ordinary algebraic problem of maxima and minima is now crystallizing into a new branch of mathematics under the leadership of Pincherle, Hadamard, Hilbert, Moore, and others. For this field Professor Moore proposes the term "General Analysis," defined (page 9) as "the theory of systems of classes of functions, functional operations, etc., involving at least one general variable on a general range." He has fixed attention on the most abstract aspect of this field by considering functions of an absolutely general variable. The nearest approach to a similar investigation is due to Fréchet (Paris thesis, 1906), who restricts himself to variables for which the notion of a limiting value is valid.

In the General Analysis we consider a class $\mathfrak{M}$ of real single valued functions ϕ_p of the variable p; important illustrative cases are:

I. $p = 1, 2, \cdots, n$; $\phi_p \equiv (\phi_1, \phi_2, \cdots, \phi_n)$, where $\phi_1, \cdots, \phi_n$ are arbitrary real quantities.

II. $p = 1, 2, \cdots, n, \cdots$; $\phi_p \equiv (\phi_1, \phi_2, \cdots, \phi_n, \cdots)$, where $\phi_1, \phi_2, \cdots$ are restricted in that $\lim_{p=\infty} \phi_p = 0$.

III. The same as II, except that the convergence restriction is now that $\sum_{p=1}^{\infty} \phi_p^2$ converges.

IV. $0 \leqq p \leqq 1$; ϕ_p is any real continuous function of p.

The class $\mathfrak{M}$ is linear (L) if the sum of every two functions of $\mathfrak{M}$ or the product of one such function by any real constant is in $\mathfrak{M}$.

The class $\mathfrak{M}$ is closed (C) if the limit ϕ_p of a convergent sequence $\alpha_p, \beta_p, \cdots$ of functions of $\mathfrak{M}$ is itself a function of $\mathfrak{M}$ whenever there exists a function λ_p of $\mathfrak{M}$ such that the difference between the successive members of the sequence and the limit ϕ_p becomes and remains uniformly not greater in absolute value than $\epsilon\lambda_p$, where ϵ is an arbitrary small positive quantity. This mode of convergence is termed relatively uniform convergence as to the scale function λ_p, and becomes uniform convergence if the scale function is a constant.

The class $\mathfrak{M}$ possesses the dominance property D if for every finite or infinite sequence $\alpha_p, \beta_p, \cdots$ in $\mathfrak{M}$ there exists a dominating function λ_p of $\mathfrak{M}$ such that each particular element of the sequence does not exceed in absolute value a suitable constant multiple of λ_p for any p.*

Finally if the absolute value of any function in the class $\mathfrak{M}$ is a function of $\mathfrak{M}$, that class is said to be absolute (A).

It is clear that in the cases I, II, III, IV the class of functions has the properties L, A and it is also easily seen that C, D will hold (cf. Theorem, page 42).

The property C in case IV is equivalent to the property that the limit of a uniformly converging sequence of continuous functions is itself a continuous function. In case II, C may be seen to hold as follows: Suppose we have a sequence of sequences each converging to zero, like terms of successive sequences converging to the like terms of a limit sequence, and furthermore in such a way that the term-by-term difference from the limit becomes and remains less than ϵ times the corresponding term of some sequence converging to zero. The limit sequence will then have the property of converging to zero since its terms are less in absolute value than the sum of the absolute values of the corresponding terms of two sequences converging to zero. In case III, C also holds and to demonstrate this fact we need merely to modify the above argument by replacing the condition 'converging to zero' by the condition 'with sum of squares convergent,' and to note that the sum of squares of the elements of the limit sequence is less than twice the sum of squares of the elements of two sequences in $\mathfrak{M}$ by virtue of the inequality

$$(a + b)^2 \leqq 2(a^2 + b^2).$$

The property D holds in case IV when we may take $\lambda_p = 1$ as the dominating function for any sequence; in case II if $M_1, M_2, \cdots$ denote positive quantities greater than any term of

* The word 'dominating' is not used in the Lectures with the same significance.

the sequence of α_p, β_p, $\cdots$ respectively in absolute value, we may take

$$\lambda_p = \frac{|\alpha_p|}{M_1} + \frac{|\beta_p|}{2M_2} + \frac{|\gamma_p|}{4M_3} + \cdots$$

since then clearly $\lim_{p=\infty} \lambda_p = 0$, so that λ_p is in $\mathfrak{M}$ and also, by definition, will dominate the members of the sequence; in case III we may take λ_p so that

$$\lambda_p^2 = \frac{\alpha_p^2}{M_1} + \frac{\beta_p^2}{2M_2} + \frac{\gamma_p^2}{4M_3} + \cdots,$$

where M_1, M_2, $\cdots$ exceed the sum of the squares for the sequences α_p, β_p, $\cdots$ respectively, since then clearly $\sum_{p=1}^{\infty} \lambda_p^2$ is a convergent series and also λ_p is a dominating function for each sequence.

Suppose now that any class $\mathfrak{M}$ of functions be given which possesses the property D. We may extend the class $\mathfrak{M}$ by adding to it all finite sums of constant multiples of functions of $\mathfrak{M}$. Since by D we may find a dominating function μ_p of $\mathfrak{M}$ for the functions in this sum, every function of this extended linear class $\mathfrak{M}_L$ does not exceed in absolute value a constant multiple of some function of $\mathfrak{M}$, for example $m\mu_p$, where μ_p is chosen as indicated and m is a sufficiently large constant. Let us now add to $\mathfrak{M}_L$ the limit functions ϕ_p of every convergent sequence of functions of $\mathfrak{M}_L$ which is relatively uniformly convergent as to a function λ_p of $\mathfrak{M}_L$; such a sequence may be written in the form of an infinite series of functions of $\mathfrak{M}_L$

$$\phi_p = \alpha_p + \beta_p + \cdots$$

since $\mathfrak{M}_L$ is linear. Inasmuch as λ_p does not exceed in absolute value some function of $\mathfrak{M}$, we may take the scale function λ_p to be in $\mathfrak{M}$. This extension of the class $\mathfrak{M}_L$ is called the $*$-extension of $\mathfrak{M}$ and is denoted by $\mathfrak{M}_*$. Since the remainder after n terms in the above series may be taken not to exceed $|\lambda_p|$ itself, by taking n sufficiently large, and since we can find a dominating function for each of these n terms and for λ_p which belongs to $\mathfrak{M}$, the same argument that was made above shows that ϕ_p does not exceed in absolute value some constant multiple of a function of $\mathfrak{M}$. Hence $\mathfrak{M}_*$ possesses the dominance property also, and the dominating function of any sequence may be taken in $\mathfrak{M}$ (cf. Theorem, page 53); it is apparent that $\mathfrak{M}_*$ is linear.

A second extension $(\mathfrak{M}_*)_*$ *adds no new functions* (cf. Theorem III, page 52). For arrange the elements of the twice extended set as a double array of functions of $\mathfrak{P}$

$$\begin{aligned}\phi_p = \ &(\alpha_p + \beta_p + \cdots)\\ &+ (\alpha'_p + \beta'_p + \cdots)\\ &+ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot,\end{aligned}$$

where the series in the successive rows converge relatively uniformly with respect to functions $\lambda_p, \lambda'_p, \cdots$ of $\mathfrak{M}$ and where the series of rows converges relatively uniformly with respect to a function $\bar{\lambda}_p$ of $\mathfrak{M}_*$. According to what has been said, the sequence $\bar{\lambda}_p, \lambda_p, \lambda'_p, \cdots$ possesses a single dominating function in $\mathfrak{M}$. All the scale functions may therefore be taken to be λ_p so that by taking a sufficiently large but finite number of elements of the double array we may obtain a function of $\mathfrak{M}_L$ which differs from ϕ_p by a quantity not exceeding $\epsilon\lambda_p$ in absolute value. Hence, we may replace the double array by a single array relatively uniformly convergent as to λ_p. This proves the statement to the effect that $\mathfrak{M}_*$ is closed under $*$-extension (cf. Theorem I, page 80).

By adding to $\mathfrak{M}$ its absolute value functions, one obtains $\mathfrak{M}_A$, and then by extending $\mathfrak{M}_A$ to be linear one obtains $(\mathfrak{M}_A)_L$; all functions obtained from $\mathfrak{M}$ by a succession of operations of this character form a class $\mathfrak{M}_{AL}$ with the properties A, L, D. The $*$-extension of $\mathfrak{M}_{AL}$ is called the #-extension of $\mathfrak{M}$ and of course has the property A in addition to L, C, D for when any sequence of functions of $\mathfrak{M}_{AL}$ is relatively uniformly convergent as to a function of $\mathfrak{M}_{AL}$ the absolute value sequence will lie in $\mathfrak{M}$ and have the same convergence property. Thus we have

$$(\mathfrak{M}_\#)_\# = [(\mathfrak{M}_\#)_{AL}]_* = (\mathfrak{M}_\#)_* = [(\mathfrak{M}_{AL})_*]_* = (\mathfrak{M}_{AL})_* = \mathfrak{M}_\#;$$

hence $\mathfrak{M}_\#$ is closed under #-extension (cf. Theorem II, page 80).

In the logical excursus of §§28–42 of Part I, Professor Moore considers the characteristic logical schemes suggested by the ideas of 'extension' and 'closure.' Thus for the subclasses $\mathfrak{N}$ containing $\mathfrak{M}$ of a fundamental class $\overline{\mathfrak{M}}$ the extension of $\mathfrak{M}$

as to P is the greatest common subclass of the classes containing $\mathfrak{M}$ and having the property P.

Part I concludes with a proof of the complete independence of the properties A, L, C, D: that is, it is proved that all the $2^4 = 16$ conceivable combinations of these properties and their negatives are consistent.

In Part II are treated the composition of classes $\mathfrak{M}$ and $\mathfrak{M}'$ on the respective ranges p and p'. If both $\mathfrak{M}$ and $\mathfrak{M}'$ possess the property D, the product class $\mathfrak{M}\mathfrak{M}'$ on the product range pp' obtained by multiplying any function ϕ_p of $\mathfrak{M}$ by any function $\phi'_{p'}$ of $\mathfrak{M}'$, has also the property D: for from any sequence $\alpha_p\alpha'_{p'}$, $\beta_p\beta'_{p'}$, $\cdots$ we deduce dominating functions λ_p for the sequence α_p, β_p, $\cdots$ and $\lambda'_{p'}$ for $\alpha'_{p'}$, $\beta'_{p'}$, $\cdots$, and thus obtain a dominating function $\lambda_p\lambda'_{p'}$ in $\mathfrak{M}\mathfrak{M}'$ for the given sequence.

We may now obtain $(\mathfrak{M}\mathfrak{M}')_*$ which is readily shown to be the same as $(\mathfrak{M}_*\mathfrak{M}')_*$, $(\mathfrak{M}\mathfrak{M}_*')_*$ or $(\mathfrak{M}_*\mathfrak{M}_*')_*$ (cf. Theorem I, page 95). In fact the classes $\mathfrak{M}_*\mathfrak{M}'$ and $\mathfrak{M}\mathfrak{M}_*'$ form part of $(\mathfrak{M}\mathfrak{M}')_*$ and contain $\mathfrak{M}\mathfrak{M}'$. Hence the statement must be true for the first two of the three classes since $(\mathfrak{M}\mathfrak{M}')_*$ is the same as its $*$-extension. By a second extension as to p' one concludes that $\mathfrak{M}_*\mathfrak{M}_*'$ belongs to $(\mathfrak{M}\mathfrak{M}')_*$ and at the same time contains of course $\mathfrak{M}\mathfrak{M}'$. Hence the statement is true for the third class. Similar results may now be given at once for the composition of more than two classes (cf. Theorems II, III, IV, pages 95, 96).

In the special case where $\mathfrak{M}$ and $\mathfrak{M}'$ are the classes of all continuous functions of p and p' respectively, the class $(\mathfrak{M}\mathfrak{M}')_*$ is the class of all continuous functions of the two variables. This well known fact serves to orient one toward the remainder of Part II. The continuity in p, p' of a function $\phi_{pp'}$ of this class $(\mathfrak{M}\mathfrak{M}')_*$ is equivalent to the following two conditions (1) that $\phi_{pp'}$ is uniformly continuous in p, *uniformly over the range* p'; (2) $\phi_{pp'}$ is continuous in p' for every p. Professor Moore treats properties B of $\mathfrak{M}$ such that for every $\mathfrak{M}'$ with properties L, C, D every function of the class $(\mathfrak{M}\mathfrak{M}')_*$ is in $\mathfrak{M}$ for every p' and has the property B uniformly, and is in $\mathfrak{M}'$ for every p (cf. §§ 57–65). Of course the precise kind of uniformity needs to be stated in each case. The fact that $(\mathfrak{M}\mathfrak{M}')_*$ in the particular case is the class of all continuous functions of pp' suggests a consideration of the class $\mathfrak{F}$ of *all* functions of pp' belonging to the class $\mathfrak{M}$ and having a property B uniformly for every p' and belonging to the class $\mathfrak{M}'$ for every p (cf. Theorem, page 109, Theorem I, page 110).

From here on, while the range p' is kept general, the range p is taken to have a certain additional property Δ closely allied to the limit property of Fréchet, but of a *non-metric* character.* In the application of the General Analysis it is found necessary to have such properties, in order to secure a satisfactory characterization of functions of $(\mathfrak{M}\mathfrak{M}')_*$ on the range pp'.†

Professor Moore has found it possible to define the analog of the class of all continuous functions over ranges of this general description including the types presented in I–IV. The property of these ranges which is generalized is that of the denumerability in I, II, III and that of sequential halving of the interval in IV.

The generalization is obtained as follows: A development Δ of the range p is a sequence $\Delta^m (m = 1, 2, \cdots)$ of systems of subclasses $\Delta^{ml} (l = 1, 2, \cdots, l_m)$ of elements of the range, for a particular m giving the mth stage Δ^m of the development Δ. A function ϕ_p dominated by a function of $\mathfrak{M}$ is said to have the property K_{12} relative to $\mathfrak{M}$ and Δ (generalization of convergence and continuity) in case the difference between the values of ϕ_p at a fixed point p^{ml} and a variable point p in any one class Δ^{ml} of the mth stage tends to zero, as m increases, relatively uniformly to some function of $\mathfrak{M}$, and at the same time ϕ_p itself tends to zero in the same manner for points p in no class Δ^{ml} of the mth stage. It is then easy to demonstrate that all functions of $\mathfrak{M}_*$ possess the property K_{12} if those of $\mathfrak{M}$ do, and also that in case $\mathfrak{M}'$ is a second class of functions having the property K_{12} with respect to $\mathfrak{M}'$ and Δ', the set $\mathfrak{M}\mathfrak{M}'$ possesses the property K_{12} with respect to the composite development $\Delta\Delta'$ (cf. Theorem II, page 135).

Suppose now that we have a set of functions of $\mathfrak{M}$, say δ_p^{ml}, one for each subset of the development Δ of the range, such that (1) the sum of the absolute value of the functions $\delta_p^{ml}\phi_{p^{ml}}$ (p^{ml} a representative point of Δ^{ml} and ϕ_p with the property K_{12}) whose corresponding Δ^{ml} *does not* contain p becomes small relatively uniformly with respect to some function ϕ_{0p} of $\mathfrak{M}$ as m increases, (2) the sum of those δ_p^{ml} whose corresponding Δ^{ml} *does* contain p, as well as the sum of their absolute values, tends toward 1 uniformly as m increases. This set of functions

* Dr. T. H. Hildebrandt deals with the matter in his Chicago thesis of 1910.

† See the article by Professor Moore in the *Atti* of the Rome International Congress, 1908, vol. 2, pp. 98–114.

have clearly the property that any such function ϕ_p may be written as

$$\lim_{m=\infty} \sum_m \phi_{pml}\delta_p^{ml}.$$

In fact we have

$$| \phi_p - \sum_m \phi_{pml}\delta_p^{ml} | \leqq | \phi_p - \sum_{m 1} \phi_{pml}\delta_p^{ml} | + | \sum_{m 2} \phi_{pml}\delta_p^{ml} |,$$

where the first and second summation are extended respectively to the functions δ_p^{ml} whose Δ^{ml} does and does not contain p. But by (1) the second term on the right is not greater than $\epsilon | \phi_{0p} |$, where ϵ is arbitrarily small for m large, and ϕ_{0p} is a function of $\mathfrak{M}$. Also the first term is not greater than

$$| \phi_p - \sum_1 \phi_p\delta_p^{ml} | + | \sum_1 (\phi_p - \phi_{pml})\delta_p^{ml} |.$$

By (2) the first member of this sum does not exceed $\epsilon|\phi_p|$ for m large if there are any terms in $\sum_1$, and by K_{12} does not exceed $\epsilon | \phi_{1p} |$ in the contrary case (p not in any Δ^{ml}). The second term does not exceed $\epsilon | \phi_{2p} |$ by K_{12} and (2) or else is zero. Hence any function ϕ_p with the property K_{12} can be represented as stated and the convergence is relatively uniform with respect to some function of $\mathfrak{M}$ (namely with respect to any function which dominates ϕ_p, ϕ_{0p}, ϕ_{1p}, ϕ_{2p}), since $\mathfrak{M}$ possesses the property D. *It follows that* ϕ_p *belongs* to $\mathfrak{M}_*$ (cf. Theorem I, page 140).

The simplest illustration of the above is afforded by II, III when

$$\delta_p^{ml} = \begin{cases} 0, p \neq l \\ 1, p = l \end{cases} \quad \text{and} \quad l = 1, 2, \cdots, m.$$

In case IV a developmental system is well known also.

The composition theory of developments Δ and of developmental systems δ_p^{ml} is now finally considered; the functions of the composite developmental systems for any stage m are given by the products of the developmental functions of each of the classes for the same stage.

The notational scheme of the General Analysis makes possible a very abbreviated statement of the results. For example the last theorem dealt with above is written

$$\mathfrak{M}^{DK_{12}\Delta} \cdot \supset \cdot \mathfrak{M}_* = [\text{all } \phi^{K_{12}\mathfrak{M}}].$$

This affirms that if $\mathfrak{M}$ possesses the properties : D, the dominance property; K_{12}, that every function of $\mathfrak{M}$ has the property K_{12} [relative to $\mathfrak{M}$ and some development Δ]; Δ, that $\mathfrak{M}$ possesses a developmental system of functions [relative to this same development Δ]; it follows that $\mathfrak{M}_*$ is the class of all functions having the property K_{12} relative to $\mathfrak{M}$ [and to Δ]. This example shows two of the principal features of the notation, namely the employment of superscripts to indicate properties, and the use of the Peano symbols.

There can be no doubt that the principal mathematical results of these lectures are of a simple character, admitting of very brief proof, and the reviewer has tried to bring out this fact. But Professor Moore has broken up his treatment into its component abstract parts, and at the same time has employed a sufficiently extensive technical notation to distinguish numerous special cases by their abstract properties; in this way a complex mathematical situation has arisen. The reason which led Professor Moore to adopt this form of treatment lies of course in the indisputable fact that the whole of mathematics needs to be presented from a standpoint which recognizes common elements of thoughts in diverse fields. It is Professor Moore who has most consistently advanced this important thesis.

The following list of *errata* has been forwarded by Professor Moore to the reviewer:

P. 128. A definition (6) is needed for use in § 74 (2). After lines 5 and 9 respectively *insert:*

$$(6)\quad \begin{aligned} &K'_{p'_1 p'_2 m} \cdot p'' \cdot \supset \cdot K_{p'_1 p'' p'_2 p'' m};\\ &p' \cdot K''_{p''_1 p''_2 m} \cdot \supset \cdot K_{p' p''_1 p' p''_2 m}. \end{aligned}$$

$$(6)\ (K'_2, K_{12})^6, (K''_2, K_{12})^6, (K'_2, K''_2, K_{12})^6;$$

P. 129, ll. 3–5. *For* final superscript ^{3} *read* ^{5}.

P. 143, ll. 16–26. From the equation

$$\theta_{mp}-\phi_p=\sum_h{}^*\phi_{r^{mh}}\delta_p^{mh}-\sum_g(\phi_p-\phi_{r^{mg}})\delta_p^{mg}+\phi_p\Big(\sum_g\delta_p^{mg}-1\Big),$$

by suitable use of (19, 12, 15, 14, 16) we obtain the relation

$$(21)\ p^m \cdot (m \geqq m_0, m_{3e}, m_{2e}, m_{4e}) \cdot \supset \cdot$$
$$A(\theta_{mp}-\phi_p) \leqq e(A\mu_{3p}+4A\mu_{2p}+aA\mu_{0p}).$$

P. 147, l. 15. *After* in fact, *insert*

$$\text{in case } \mathfrak{M}'^{D_1K'_2} \text{ and } \mathfrak{M}''^{D_1K''_2},$$

P. 147, ll. 20–21. Insert the respective conditions

$$D_1K'_2,\ D_1K''_2;\ K'_2,\ K''_2,$$

on the classes $\mathfrak{M}'$, $\mathfrak{M}''$ in the hypotheses of the propositions of lines 20; 21.

P. 148, l. 12. *For* $D\Delta$ *read* $DK_2^{\Delta}\Delta$.

P. 148, l. 25. *For* $LCD\Lambda$ *read* $LCDK_2^{\Delta}\Delta$.

II. *Projective Differential Geometry.*

In these lectures Professor Wilczynski presented an outline of some of the most important results of the projective differential geometry of curves and ruled surfaces, and at the same time gave an indication of the method of proof. A complete presentation of these results will be found in Professor Wilczynski's treatise on Projective Differential Geometry. For this reason the reviewer confines himself to a very brief synopsis of these readable lectures.

Projective differential geometry deals with the differential properties of geometrical configurations that are invariant under the projective group.

The curve in $(n-1)$ space presents itself as given by n homogeneous coordinates $y_1, \cdots, y_n$, each a function of the parameter x. Any other set of such coordinates in the same parameter will be given by any n linearly independent combinations of these, aside from a multiplicative transformation. Hence the homogeneous linear differential equation of the nth order, a fundamental set of whose solutions is $y_1, \cdots, y_n$ is the same as that given by any other projectively equivalent curve, or else is obtained from it by a multiplicative transformation of the dependent variable. If the factor is chosen so as to make the $(n-1)$th derivative disappear, the ratios of the remaining coefficients will then be projective invariants of a given curve with given parameter and are called the semi-invariants. The semi-covariants are similarly defined.

The absolute invariants and covariants are further invariant and covariant under a transformation of the parameter and are readily obtainable from the semi-invariants and covariants.

The geometrical interpretation of these functions leads to the projective differential geometry of curves, and is not given in the lectures by Professor Wilczynski. His contribution in this direction has been to make systematic use of the elegant analytic instrument afforded by the differential equation, and thus to obtain Halphen's results for the plane and space, and in addition like results in a general space.

The projective theory of the ruled surfaces in space whose generators are the lines joining corresponding points of two curves C_y and C_z with coordinates

$$y_1, y_2, y_3, y_4 \text{ and } z_1, z_2, z_3, z_4$$

is then treated. The most general set of coordinates (aside from a factor) is obtained by taking the same four linearly independent linear combinations of the y's and z's. Thus we are led to consider

$$\sum c_i y_i \text{ and } \sum c_i z_i.$$

These functions form the general solution of a pair of ordinary linear differential equations

$$y'' + p_{11}y' + p_{12}z' + q_{11}y + q_{12}z = 0,$$
$$z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z = 0.$$

Conversely any set of four linearly independent particular solutions y, z of such equations yield always projectively equivalent ruled surfaces. These equations are fundamental for the theory which has been created by Professor Wilczynski.

A change of the dependent variables

$$\bar{y} = \alpha(x)y + \beta(x)z, \quad \bar{z} = \gamma(x)y + \delta(x)z$$

changes C_y and C_z into any other pair of curves $\bar{C}_{\bar{y}}$ and $\bar{C}_z$ on the same ruled surface, and a change of independent variable $\bar{x} = f(x)$ changes the parameter on the curves in any given way. The invariants and covariants under these transformations are now obtained analytically and their interpretation leads to the geometrical results.

Since the space dual of a ruled surface is a second ruled surface one may expect a second pair of adjoint equations to be associated with the given one. The invariants are the same for the adjoint equations as for the given equations, except in sign. The condition that these two sets of equations are the same leads to the theorem that a ruled surface is projectively equivalent to its dual only in case it is a quadric.

The asymptotic lines on ruled surfaces are clearly projectively invariant. The condition that the curves C_y and C_z are asymptotic lines is simply $p_{12} = p_{21} = 0$ and in this case the integral curves of the given equations form the asymptotic lines. A simple consequence is the theorem of Paul Serret: the cross ratio formed by the intersection of a moving generator with four fixed asymptotic lines is constant.

The totality of the tangents to the asymptotic lines along a given generator g form one set of rulings of an osculating hyperboloid of the ruled surface. The totality of generators of the hyperboloids thus constructed form a congruence Γ. One of the covariants P gives a unique second generator g' of the hyperboloid H of the same set as g and thus a second ruled surface varying arbitrarily with the choice of independent variable of which it serves as the image. This ruled surface is the derivative ruled surface S' of S with respect to x.

The flecnode curves are two curves along which the tangents to the asymptotic lines have four points in common with the ruled surface. The flecnode points are determined by the equating to zero of a certain quadratic covariant C. If the flecnode points coincide, one of the invariants θ_4 vanishes. The developables of which the flecnode curves are the cuspidal edges form the focal surfaces of the congruence Γ. The condition that another invariant Δ vanishes is that the ruled surface belongs to a linear complex.

In the final lecture there are given first some theorems concerning the derivative surface of S. The lecturer then passes on to develop the notion of the two complex points (arising from another quadratic covariant) on each generator which with the flecnode points form a harmonic group. Lastly the theorems which state the extent to which the flecnode curves may be arbitrarily assigned are given.

III. *Theory of Boundary Value Problems.*

Professor Mason considers a functional equation $f = g + Sf$ for x on a range R, where S is a linear operator such that (1) $S\phi$ is continuous when ϕ is continuous, (2) the series

$$\phi + S\phi + S^2\phi + \cdots$$

converges uniformly in R, and (3) the result of the operation S on the series is the same as operating with S term by term. In this case S is suggestively termed a convergent operator and the infinite series above for $\phi = g$ is easily proved to form the unique continuous solution of the given equation.

As a first application, the existence theorem for ordinary linear differential equations of the second order is obtained. The method does not differ from the method of successive approximation, save in form.

The second application is to a linear partial differential equation in two independent variables of hyperbolic type. The standard reduction of linear partial differential equations of the second order to normal forms is first effected. In the hyperbolic case, this form is

$$\frac{\delta^2 u}{\delta x \delta y} = a\frac{\delta u}{\delta x} + b\frac{\delta u}{\delta y} + cu + f.$$

Professor Mason treats the following new boundary value problem: Given a rectangle $x_1 \leqq x \leqq x_2$ and $y_1 \leqq y \leqq y_2$ and upon it two curves $C_x : y = \phi(x)$ $(x_1 \leqq x \leqq x_2)$ and $C_y : x = \psi(y)$ $(y_1 \leqq y \leqq y_2)$ where ϕ, ψ, and the derivative of ψ are continuous, to determine a solution u of the above equation such that u and $\delta u/\delta y$ take assigned values $U(x)$ and $Y(y)$ along C_x and C_y respectively. Professor Mason proves that a unique solution u exists. The critical portion of this demonstration is less satisfactorily presented than the remainder of the lectures. The operator S which appears in this case is shown to admit of a "majorant operator."

This boundary value problem contains most of those already considered for this type of equation, including the classical case in which the curves reduce to $x = x_0$ and $y = y_0$ respectively and that in which u and its normal derivative are given along one and the same monotonic curve.

The simplest equation of elliptic type, which is the potential equation in the plane, is next considered, and the method of Neumann is employed to show that a unique solution exists for a convex region bounded by a regular curve $x = \xi(t)$, $y = \eta(t)$ which takes given values along the boundary. In this case the operator S is

$$S(\phi) = \frac{1}{\pi}\int_0^l \left\{ [\phi(s) - \phi(t)] \frac{\delta}{\delta t} \text{arc tan} \frac{\eta(s) - \eta(t)}{\xi(s) - \xi(t)} \right\} dt,$$

which is also proved to be convergent. A short review of the more general methods which permit one to consider regions not convex is given.

The Green's function $G(x, y : \xi, \eta)$ for the potential equation is exhibited in the customary way by means of the result just stated. Then follows the interesting question of the existence of doubly periodic solutions u of $\Delta u = f(x, y)$, where $f(x, y)$ is doubly periodic in x, y with periods a, b. The real part of log $\sigma(z - \xi) - \log \sigma(z - \eta)$, where σ is the Weierstrassian σ-function and $z = x + iy$, furnishes a solution of $\Delta u = 0$ analytic at all points of the period rectangle save for an infinity at $z = \xi$ and $z = \eta$ like $\pm \log r$ (r the distance from x, y to ξ, η) and furthermore is doubly periodic except for simple linear terms. By a slight modification destroying these terms the Green's function for the equation $\Delta u = 0$ and the boundary conditions $u(\alpha) = u(\alpha + a)$, $u(\beta) = u(\beta + b)$ are obtained. The necessary and sufficient condition for a doubly periodic solution is found by Professor Mason to be that the integral of f over the rectangle vanishes.* The proof depends on the properties of the Green's function.

It is next shown that a solution of

$$\Delta u + cu = f$$

taking assigned values on the boundary of a region R exists if R is taken sufficiently small. The operator S is obtained by means of the Green's function of $\Delta u = 0$.

Attention is then called by the lecturer to the recent work of Serge Bernstein, in which are proved simple criteria for the analytic or non-analytic nature of solutions of partial differential equations of the second order.

In conclusion, the lecturer considers the boundary value problem attaching to the equation and condition

$$y'' + \lambda A(x)y = 0, \quad y(a) = y(b) = 0, \quad (\lambda \text{ a parameter}),$$

as it arises from the consideration of the transverse vibrations of a stretched string. The method of the existence proof depends on the solution of a certain minimum problem, and in order to

* See an article in the *Transactions of the American Mathematical Society*, vol. 6 (1905), p. 159.

carry through this proof the general solution of

$$y'' + \lambda A(x)y = f, \quad y(a) = y(b) = 0$$

is first derived. The minimum problem which is to be treated is to minimize $\int_a^b y'^2 dx$ under the condition that $\int_a^b Au^2 dx = 1$ and $\int_a^b y_i y dx = 0$ $(i = 1, 2, \cdots, n)$, where $y_1, \cdots, y_n$ are the solutions belonging to $\lambda_1, \cdots, \lambda_n$. This gives in order of increasing magnitude the positive values of λ; these exist in infinite number if ϕ is anywhere positive. Likewise a series of negative values of λ will be obtained if ϕ is anywhere negative.*

A formal expansion of an arbitrary function is then given by

$$f = \sum_{-\infty}^{+\infty} c_i y_i, \quad c_i = \pm \int_a^b f A_i y_i dx,$$

and Professor Mason states the theorem that this expansion holds if f vanishes at a and b, is continuous, and has a derivative continuous save at a finite number of points. The proof however contains an error.†

G. D. BIRKHOFF

SHORTER NOTICES.

Vorlesungen über Algebra. Von GUSTAV BAUER. Herausgegeben vom Mathematischen Verein München. 2te Auflage. Leipzig und Berlin, B. G. Teubner, 1910. vi + 366 pp.

THAT Professor Bauer's lectures, which were published in honor of his 80th birthday by the Mathematical Club of Munich in 1903, are destined to outlive their author by many years, seems to be evidenced by the fact that a second edition became necessary in 1910, seven years after the first edition and four years after Professor Bauer's death, which occurred on April 3, 1906.

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*See an article in the *Transactions*, vol. 8 (1907), p. 373.

† At bottom of p. 219 it is necessary to replace the multiplier 2 in the inequality

$$(\phi_{n,m})^2 \leqq 2 \sum_{i=n}^{m} c_i^2 \left(\int^x y_i' dx \right)^2$$

by a multiplier $m - n$. This error appears to destroy the force of the proof.