THE SEVENTEENTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE seventeenth annual meeting of the Society was held in New York City on Wednesday and Thursday, December 28-29, 1910, extending through the usual morning and afternoon sessions on each day. The attendance included the following sixty-three members of the Society:

Mr. F. W. Beal, Professor W. J. Berry, Professor G. D. Birkhoff, Professor Maxime Bôcher, Professor Joseph Bowden, Dr. Emily M. Coddington, Professor A. B. Coble, Professor F. N. Cole, Dr. J. R. Conner, Dr. G. M. Conwell, Professor J. L. Coolidge, Professor E. W. Davis, Professor F. F. Decker, Dr. L. S. Dederick, Professor W. P. Durfee, Professor L. P. Eisenhart, Professor J. C. Fields, Professor H. B. Fine, Professor W. B. Fite, Professor A. S. Gale, Professor O. E. Glenn, Professor C. C. Grove, Professor C. O. Gunther, Professor E. L. Hancock, Professor J. G. Hardy, Professor C. N. Haskins, Professor H. E. Hawkes, Dr. W. A. Hurwitz, Professor L. A. Howland, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Dr. D. D. Leib, Dr. N. J. Lennes, Mr. Joseph Lipke, Professor W. R. Longley, Mr. A. R. Maxson, Professor Helen A. Merrill, Dr. H. H. Mitchell, Professor C. L. E. Moore, Professor C. .N Moore, Professor Frank Morley, Professor Richard Morris, Professor G. D. Olds, Dr. H. B. Phillips, Dr. H. W. Reddick, Professor R. G. D. Richardson, Miss S. F. Richardson, Mr. L. P. Siceloff, Mr. C. G. Simpson, Dr. Clara E. Smith, Mr. F. H. Smith, Professor P. F. Smith, Mr. W. M. Smith, Dr. W. M. Strong, Dr. M. O. Tripp, Professor C. B. Upton, Professor Oswald Veblen, Mr. H. E. Webb, Professor H. S. White, Mr. E. E. Whitford, Professor E. B. Wilson, Professor T. W. D. Worthen.

President Maxime Bôcher took the chair at the opening session, yielding it later to Professor Kasner, Ex-President Osgood, and at the morning session on Thursday to the President elect, Professor H. B. Fine. The Council announced the election of the following persons to membership in the Society: Professor Percy Hodge, Columbia University; Mr. C. G. P. Kuschke, University of California; Professor Marion B. White, University of Kansas. Eight applications for membership were received.

On Wednesday evening thirty of the members gathered at the annual dinner, which has for many years been a pleasant adjunct to the regular sessions.

It was decided to hold the summer meeting in 1911 at Vassar College. Professors H. S. White, P. F. Smith, and the Secretary were appointed a committee to make the necessary arrangements. The summer meeting of 1912 will be held at the University of Pennsylvania.

The report of the Treasurer, Auditing Committee, and Librarian are published in the Annual Register. The membership of the Society has increased during the past year from 618 to 641, including at present 60 life members. The total attendance of members at all meetings was 317. The number of papers presented was 145. In the annual election 202 votes were cast. The Society's library shows the normal yearly increase, the total number of books on the shelves being now 3,508. The Treasurer's report shows a balance of $8,124.53, a slight increase over last year, although the subventions received from various universities for the support of the *Transactions* have been discontinued, and the Society has paid a first installment of 1,000 francs of its contribution toward the publication of the collected works of Euler. The income from sales of the Society's publications during the year was $1,660.21. The life membership fund now amounts to $3,901.83.

During the past sixteen years the Society has expended for printing the BULLETIN, *Transactions*, and other publications $42,473.57. The total returns from publications have been in the same period $15,375.20. The university subventions have amounted to $8,100. Editorial and administrative expenses have been $11,779.49. The stock of publications on hand may be fairly valued at about $10,000.

At the annual election, which closed on Thursday morning, the following officers and other members of the Council were chosen:

President,	Professor H. B. FINE.
Vice-Presidents,	Professor G. A. BLISS, Professor W. E. STORY.
Secretary,	Professor F. N. COLE.
Treasurer,	Professor J. H. TANNER.
Librarian,	Professor D. E. SMITH.

Committee of Publication,
Professor F. N. COLE,
Professor E. W. BROWN,
Professor VIRGIL SNYDER.

Members of the Council to serve until December, 1913,
Professor H. F. BLICHFELDT, Professor C. J. KEYSER,
Professor J. L. COOLIDGE, Professor J. W. YOUNG.

The following papers were read at this meeting:

(1) Mr. S. CHAPMAN: "A note on the theory of summable integrals."

(2) Dr. H. H. MITCHELL: "Concerning a rotation group in six-space."

(3) Professor VIRGIL SNYDER: "An application of a (1–2) quaternary correspondence."

(4) Rev. A. S. HAWKESWORTH: "Three new dimension theorems."

(5) Professor W. R. LONGLEY: "Singular points on the discriminant locus of an ordinary differential equation."

(6) Professor R. G. D. RICHARDSON: "Theorems of oscillation for two self-adjoint linear differential equations of the second order with two parameters."

(7) President MAXIME BÔCHER: "A simple proof of a fundamental theorem in the theory of integral equations."

(8) Professor J. L. COOLIDGE: "The metrical aspect of the line-sphere transformation."

(9) Dr. E. J. MILES: "The absolute minimum of a definite integral in a special field."

(10) Professor J. C. FIELDS: "A method of proving certain theorems relating to rational functions which are adjoint to an algebraic equation for a given value of the independent variable."

(11) Professor F. F. DECKER: "Concerning the order of a restricted system of equations."

(12) Professor PAUL SAUREL: "On the classification of crystals."

(13) Professor A. B. COBLE: "An application of Moore's cross ratio group to the solution of the sextic equation."

(14) Professor A. B. COBLE: "The cubic surface and plane six-point."

(15) Professor C. L. E. MOORE: "Conjugate directions on a hypersurface in S_4 and some allied curves."

(16) Professor W. H. BATES: "An application of symbolic methods to the treatment of mean curvature in hyperspace."

(17) Dr. H. B. PHILLIPS: "The Galois theory of sets of multipartite variables."

(18) Dr. J. R. CONNER: "Correspondences associated with the rational plane quintic curve."

(19) Professor L. P. EISENHART: Conjugate systems and envelopes of spheres."

(20) Mr. JOSEPH LIPKE: "Natural families of curves in a general curved space of n dimensions."

(21) Professor O. E. GLENN: "On the discriminants of ternary forms."

Mr. Chapman's paper was presented to the Society and read by Professor C. N. Moore. In the absence of the authors, the papers of Professor Snyder, Mr. Hawkesworth, Dr. Miles, Professor Saurel, and Professor Bates were read by title. Professor Bates's paper has appeared in full in the January *Transactions*. Professor Saurel's paper will be published in the BULLETIN. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. The main result of Mr. Chapman's paper is a proof of the following theorem: If the integral

$$\int_a^\infty f(x)dx$$

is summable (C_r), i. e., if

$$\lim_{x=\infty} \frac{r\,!}{x^r}\int_a^x \int_a^{\alpha_1} \int_a^{\alpha_2} \cdots \int_a^{\alpha_r} f(\theta)d\theta d\alpha_r \cdots d\alpha_2 d\alpha_1$$

exists, and if furthermore $f(x)/x^r$ is uniformly continuous in the interval $x \geqq k > 0$, then $\lim_{x=\infty} f(x)/x^r = 0$.

For the case $r = 1$, this theorem reduces to one due to C. N. Moore,* except that the condition in the latter theorem that $f(x)$ should be uniformly continuous is replaced by the less stringent condition that $f(x)/x$ should have that same property.

2. A collineation group in ordinary three-space, when represented on the Plücker line coordinates, is a linear group in six variables with a quadratic invariant. If a transformation be effected so that the quadratic invariant becomes

$$\sum_{i=1}^{6} x_i^2 = 0,$$

* Cf. *Transactions*, vol. 8 (1907), p. 314.

the group becomes one of orthogonal substitutions, and may be regarded as a group of rotations about a point in six-dimensional space.

Groups of order 2520 and 11520 in three-space correspond to groups leaving invariant regular solids in six-space. The group of order 25920 has not this property, but is shown by Dr. Mitchell to leave invariant a semi-regular solid of 27 vertices, bounded by 27 regular five-dimensional solids of 10 vertices and 72 regular five-dimensional solids of 6 vertices.

3. The (1, 2) correspondence discussed by Professor Snyder is defined by a triply infinite system of quadric surfaces passing through six basis points. The surfaces of this system passing through a seventh point P will also pass through a second point Q. A three-dimensional involution I exists between P and Q. If the parameters in the equation of the quadrics are regarded as plane coordinates in a second space, a (1, 2) point correspondence between the two spaces is established. The surface of branch points in the second space is a general Kummer surface, and the locus of coincident points in the first is the Weddle surface. A number of birational transformations are discussed and the conditions that the basis points are in one, two, $\cdots$, six fold involution are determined.

4. Mr. Hawkesworth presented the three following dimension theorems:

(1) The number of A-dimensional boundaries of an N-dimensional rectangular figure is

$$\frac{N!}{(N-A)!\,A!}\cdot 2^{N-A}.$$

(2) The number of A-dimensional boundaries of an N-dimensional tetrahedroidal figure is

$$\frac{(N+1)!}{(N-A)!\,(A+1)!}.$$

From this it follows that the numerical values of the boundaries must rise and fall symmetrically to a medial value, the number $N+1$ of the vertices equalling that of the last $N-1$ boundaries, the edges and the $N-2$ boundaries being alike $\frac{1}{2}(N+1)\,N$, and the planes and the $N-3$ boundaries each being $(N+1)\,N\,(N-1)/3\,!$, and so on.

These two formulas hold true, even when A is taken greater than N, the negative and reciprocal result, as for example that there are $\frac{1}{6}$ cubes in a square, meaning simply that, reversely and reciprocally, there are 6 squares in a cube.

(3) In any N dimensional rectilinear figure where N is even the boundaries, taken in their sequence alternately plus and minus (i. e., + vertices — edges + planes — solids + $\cdots$), always sum up to zero. But if N be an odd number, then the summation gives + 2.

Or, if we count the N-dimensional figure as one of its own boundaries, and thus add — 1 or + 1 to the summation, according as N is odd or even, then the summation is always + 1 for all rectilinear figures. In this case the binomial development of $(2 - 1)^N$ will give us both the sequence and the summation of the boundaries of a rectangular N-dimensional figure, while in a similar way the symmetrical sequence and summation of those of an N-dimensional tetrahedroidal figure can be represented by $-(1 - 1)^{N+1}$, omitting the first term of the binomial development.

5. The discriminant of an ordinary differential equation of the second degree, $Ay'^2 + 2By' + C = 0$, is $\Delta = B^2 - AC$. It is supposed that, within the region considered, the coefficients are analytic functions of x and y, and that $y = \eta(x)$ is an analytic branch Γ of the discriminant locus $\Delta = 0$. If the points at which both A and C vanish are excluded, the nature of the solutions in the neighborhood of any point on Γ is known. The object of Professor Longley's paper is an investigation of the integral curves through a point P on Γ at which A, B, and C vanish. It is found that an infinite number of integral curves may pass through P and the nature of these curves is not determined by the nature of Γ. For example, Γ may be a singular solution. Then through every point except P there passes one other integral curve, and it is tangent to Γ; while an infinite number of integral curves may pass through P no one of which is tangent to Γ. Or Γ may be a cusp locus, so that through every point except P there passes one integral curve, and it has a cusp at this point; while an infinite number of integral curves may pass through P no one of which has a cusp at P. In this case it may happen also that for any point except P the integral curve is not tangent to Γ, while at P every one of an infinite number is tangent to P.

6. Klein has considered the problem of determining the two parameters of the differential equation of Lamé in such a manner that the equation furnishes two solutions, one of which vanishes at the end points of a certain interval a_1b_1 and oscillates m times within the interval, and the other vanishes at the end points of a second interval a_2b_2 and oscillates n times in the interval, m and n being positive integers or zero arbitrarily prescribed. Other special problems of a special nature which involve two or more parameters have been treated by Klein and Bôcher.* For the case of two parameters these may be considered as special cases of the following: If $p_i(x) > 0$, $q_i(x) \leqq 0$, $A_{ij}(x), (i = 1, 2; j = 1, 2)$ are given analytic functions of x, when can the parameters λ, μ be so determined that the equations

$$(p_1u')' + q_1 u + (\lambda A_{11} + \mu A_{12}) u = 0, \quad (p_2v')' + q_2v + (\lambda A_{21} + \mu A_{22})v = 0$$

have solutions $u(x)$, $v(x)$ which satisfy the boundary conditions

$$u(a_1) = u(b_1) = 0, \quad v(a_2) = v(b_2) = 0$$

and oscillate m and n times within the intervals a_1b_1 and a_2b_2 respectively (the intervals a_1b_1 and a_2b_2 may or may not coincide)? By transforming the two equations into a partial differential equation with one parameter, Hilbert showed in a paper read before the Mathematische Gesellschaft zu Göttingen that in case

$$p_1 = p_2 = 1, \; q_1 = q_2 = 0, \; A_{12} = A_{21} = 1, \; A_{12} > 0, \; A_2 < 0$$

there exist an infinite number of parameter pairs $\lambda_1, \mu_1; \lambda_2, \mu_2;$ . . . for which solutions exist.

Professor Richardson has attacked the problem from the standpoint of the calculus of variations and shows that in case $A_{12} > 0$, $A_{22} < 0$, and $A_{11}(x) A_{22}(x) - A_{12}(x)A_{21}(x)$ is not identically zero there exist parameter values λ, μ such that the solutions $u(x)$, $v(x)$ oscillate m and n times respectively. He shows further that the oscillation theorem holds true when the intervals coincide and A_{12} and A_{22} have the same sign throughout the interval while $A_{11} A_{22} - A_{21} A_{12}$ is different from zero. The results hitherto obtained for problems of this type are special cases of these theorems.

7. The theorem considered in Professor Bôcher's paper states that a homogeneous linear integral equation of the second kind can have only a finite number of linearly independent

*Cf. article by Bôcher, Encyklopädie der mathematischen Wissenschaften, II A 7a.

solutions, and gives an explicit formula bounding this number. The proof depends only on the well known fact that the Gramian of a set of functions is positive or zero according as these functions are linearly independent or dependent.

8. The line-sphere correspondence of Lie owes its importance largely to the fact that it is a contact transformation, intersecting lines corresponding to tangent spheres. In Professor Coolidge's paper it is shown that this is merely a special case of a more general relation connecting distances of lines and angles of spheres. Applications are made to the linear complex and congruence, and the dual projective geometry of two ternary domains. The paper has been published in the January *Transactions*.

9. In Dr. Miles's paper the absolute minimum of a definite integral

$$J = \int_{t_0}^{t_1} F(x, y, x', y')dt$$

in a special field is discussed. Under the assumptions ordinarily imposed on the function F and the admissible curves it is known that there exists a one-parameter family of extremals passing through the point 0. This family has in general an envelope and it is here supposed that the envelope has a cusp. The covering of the region near the cusp point is considered. It is shown that outside the V-shaped cusp region there is a single extremal through each point, while for points within there are three, one of which has already touched the envelope. Then the field about an extremal arc whose end points are conjugate is discussed. The assumption is made that the conjugate point is a cusp point in the envelope, and it is found that the field is a three-sheeted affair similar to a Riemann surface, the sheets being joined by the two branches of the envelope. Further the envelope law holds along either branch of the envelope. A curve M in the V-shaped cusp region is then determined which marks the cessation of absolute minimum. This curve is continuous, passes through the cusp point 1, and is such that the extremal ceases to furnish an absolute minimum as soon as it meets this curve. Finally sufficient conditions for an absolute minimum in this field are considered.

10. In Professor Fields' paper proofs were given of the following theorems:

(1) The degree of a rational function which is adjoint to an algebraic equation for the value $z = \infty$ must be less than the degree of the equation written in the form

$$v^n + F_{n-1}v^{n-1} + \cdots + F_0 = 0,$$

and the degree of the element of the function involving the power v^{n-1} must be $\leqq n - 1$.

(2) In the case of an integral algebraic equation, an integral rational function must be divisible by the factor $z - a$ if its order of coincidence with the n branches corresponding to the value $z = a$ severally exceed by 1 the orders of coincidence defining adjointness for these branches.

(3) If an integral rational function is adjoint for $n - 1$ of the n branches of an integral algebraic equation corresponding to a value $z = a$, it must also be adjoint for the remaining branch.

The principle employed in the proof of each of the above theorems is the same. It is possible in a certain manner to construct rational functions which for a given value of the variable z possess any desired set of adjoint orders of coincidence. In the case of a finite value $z = a$ these special functions are integral. In any case they conform to the requirements of the theorem in question. Now if there exists a function which fails to accord with one of the theorems, the sum obtained on adding to this function one of the corresponding special functions will also fail to accord with the theorem. Proper selection however of the function to be added would, in the case of either of the first two theorems, give a sum whose orders of coincidence with the n branches are indefinitely great, while in the case of the third theorem the orders of coincidence of the corresponding sum with $n - 1$ of the branches would be indefinitely great. Such orders of coincidence are readily shown in each case to be compatible only with the requirements of the theorem in question. This however is in contradiction to our first deduction that the sum of a function which does not accord and a function which does accord with one of the theorems does not itself accord with the theorem. It follows therefore that no function to which any one of the theorems has reference fails to conform to that theorem.

11. Professor Decker's paper establishes a method for the determination of the order of the restricted system of equations which arise from putting equal to zero the determinants obtained

by suppressing each set of $n-m$ columns of a matrix of m rows and n columns. The method of treatment is based on the linear independence of certain of the determinants and the expression of the others in terms of them. The apparent order of the original system is found to depend upon that of a second and this in turn upon that of a third, etc., until finally a reduction formula is established by means of which the determination of the order is made to depend upon the orders of determinants.

The order is calculated for certain cases. These results specialized with regard to the degree of the elements compared with Segre's specialized with regard to the order of the determinants are found to agree. They also confirm the statement of Salmon.

13. The object of Professor Coble's first paper is to estimate the utility of E. H. Moore's cross ratio group for the solution of the sextic. The group appears as the natural bond between the given sextic and a resolvent sextic of the general diagonal type. A material algebraic advantage is gained by using this resolvent rather than the given sextic.

14. In Professor Coble's second paper some formulas of the first are employed to obtain an equation of a cubic surface mapped from a plane by means of cubic curves on a given six-point. The equations of the forty-five triple tangent planes of the surface are derived. An interesting further result is a complete system of invariants for the plane six-point.

15. Two consecutive tangent S_3's to a hypersurface in S_4 will intersect in a plane which is tangent to the hypersurface. This plane and the line joining the points of contact are said to be conjugate. In this paper Professor Moore has discussed this correspondence, also the correspondence between the line of intersection of three consecutive tangent S_3's and the plane determined by the points of contact.

17. In the usual treatment of the Galois theory of equations the property of $x_1, \cdots, x_n$ of being roots of an algebraic equation in one variable is not essential. These quantities may therefore be generalized without affecting the validity of many results in that theory. In the paper of Dr. Phillips they are considered as multipartite numbers representing points in a plane or space of higher dimensions.

18. Making use of the properties of the rational norm-curve in a space of five dimensions, Dr. Conner proves the existence of certain multiple correspondences between the planes of the two curves $\rho x_i = (\alpha_i t)^5$; $\rho \eta_i = (b_i t)^5$ $(i = 0, 1, 2)$, where the b's are three linearly independent forms apolar to the α's. The significance of these correspondences for the general rational plane quintic curve is then discussed.

19. Darboux has considered two parameter families of spheres which possess the property that the lines of curvature on the two sheets of the envelope correspond, and he has shown that the surface of centers of the spheres is cut in a conjugate system by the developables of the congruences of normals to the two sheets. Guichard has called a conjugate system of this sort a "reseau 20." Professor Eisenhart considers the determination of these systems on any surface. It is reducible to the solution of a system of linear partial differential equations of the first order. All conjugate systems with the same Gaussian representation as a "reseau 20" are likewise of this type, and with one of them there is associated a cyclic system whose circles pass through a point. This unique surface may be taken as characteristic of the group. Several interesting particular cases are discussed, one of them leading to results concerning surfaces of Monge.

20. In a paper read at the Princeton summer meeting of the Society, Mr. Lipke gave the complete geometric characterization of natural families of curves on a general surface in euclidean space of three dimensions. Natural families of curves are defined as the systems of extremals connected with variation problems of the type $\int F ds = minimum$, where ds is the element of arc length in the space considered, and F is any point function. It is the purpose of the present paper to give the complete geometric characterization of natural families of curves on a general hypersurface V_n in a euclidean S_{n+1}. The characteristic properties are: (1) the locus of the centers of geodesic curvature of the ∞^{n-1} curves passing through a point is a euclidean S_n; (2) the n directions in which the osculating geodesic circles (circles of constant geodesic curvature) hyperosculate are mutually orthogonal. The natural families are a special type of a much larger class of curves which are characterized by property (1). We find that our two-dimensional results are included in our n-dimensional results.

21. The ordinary discriminant of the ternary quadratic form $a_x^2 + b_x x_3 + c x_3^2$ is expressible as the resultant of the binary forms a_x^2, b_x plus c times the discriminant of a_x^2. Professor Glenn proves the corresponding general theorem for the ternary form of order m, and derives explicit formulas for the $\frac{1}{2}m$ $(m - 1)$ ordinary discriminants (conditions that the forms degenerate into distinct linear factors) in terms of the coefficients of the form. One of these discriminants is a linear expression in the resultant of a_x^m and b_x^{m-1} and the discriminant of a_x^m, while the rest are obtained from this one by operating by powers of

$$\Delta_1 = ma_0 \frac{\partial}{\partial b_0} + (m - 1)a_1 \frac{\partial}{\partial b_1} + \cdots + a_{m-1} \frac{\partial}{\partial b_{m-1}}$$

and

$$\Delta_2 = ma_m \frac{\partial}{\partial b_{m-1}} + (m - 1)a_{m-1} \frac{\partial}{\partial b_{m-2}} + \cdots + a_1 \frac{\partial}{\partial b_0}.$$

He also considers the explicit form of the satellite form of the m-ic and its character as an invariant.

F. N. COLE,
Secretary.

THE WINTER MEETING OF THE CHICAGO SECTION.

THE twenty-eighth regular meeting of the Chicago Section of the American Mathematical Society was held at the University of Minnesota, Minneapolis, Minn., on Wednesday, Thursday, and Friday, December 28–30, 1910, in affiliation with the American Association for the Advancement of Science. Five half-day sessions were held, beginning Wednesday afternoon with a meeting in connection with Section A (mathematics and astronomy) to hear the address of the retiring vice-president, Professor E. W. Brown of Yale University, on "The relations of Jupiter with the asteroids." Other papers of a mathematical character at this session and at its continuation on Thursday morning were by Professor F. R. Moulton of the University by Chicago on "The debt of mathematics to astronomy," and of Dr. W. D. Macmillan of the University of Chicago on "An integrable case in the problem of three bodies." The vice-president of Section A, Professor E. H. Moore of the University of Chicago, presided throughout this session.