THE FOURTH REGULAR MEETING OF THE SOUTHWESTERN SECTION.

THE fourth regular meeting of the Southwestern Section of the Society was held at the University of Nebraska, Lincoln, Nebraska, on Saturday, November 26, 1910. The following members of the Society were present:

Professor W. C. Brenke, Professor E. W. Davis, Professor E. P. R. Duval, Professor C. C. Engberg, Professor E. R. Hedrick, Professor O. D. Kellogg, Miss H. H. MacGregor, Professor W. H. Roever, Professor O. Schmiedel, Professor J. W. Young.

The morning session was opened at 10 A. M. and the afternoon session at 2 P. M., Professor Davis presiding. St. Louis was fixed as the next place of meeting of the section, and the following program committee elected: Professor Roever (chairman), Professor Brenke, Professor Kellogg (secretary). On Friday evening a smoker was held at the home of Professor Davis, at which some of the problems of the mathematical departments of the state universities were informally discussed, and the attending members lunched together on Saturday.

The following papers were presented at this meeting:

(1) Professor G. A. MILLER: "Abstract definition of all the substitution groups whose degrees do not exceed seven."

(2) Dr. ELIZABETH R. BENNETT: "Primitive groups of degree 20."

(3) Professor W. C. BRENKE: "On the series of zonal harmonics."

(4) Professor A. B. FRIZELL: "On certain transfinite permutations."

(5) Professor A. D. PITCHER: "Properties of certain classes of sequences."

(6) Professor E. W. DAVIS: "Imaginaries on a cubic" (preliminary communication).

(7) Dr. LOUIS INGOLD: "Curves in a function space."

(8) Dr. LOUIS INGOLD: "Surfaces in a function space."

(9) Professor FLORIAN CAJORI: "Horner's method of aproximation anticipated by Ruffini."

(10) Professor W. H. ROEVER: "Southerly deviation of falling bodies. Second paper."

(11) Professor J. W. YOUNG: "On fundamental regions for cyclical groups of linear transformations on two complex variables."

(12) Professor O. D. KELLOGG: "Green's integral for multiply connected regions."

(13) Professor E. R. HEDRICK: "On assemblages with closed derivatives."

Dr. Bennett was introduced by Professor Davis. The papers of Professors Miller, Frizell, and Pitcher were read by title. Dr. Ingold's papers were presented by Professors Kellogg and Hedrick, respectively, and Professor Cajori's paper by Professor Davis. Abstracts of the papers follow below.

1. In addition to giving a list of possible abstract definitions of the substitution groups whose degrees do not exceed seven, and an explanation of these and other possible abstract definitions, Professor Miller proves several general theorems relating to such definitions. Among these are the following: If n operators of order two are such that the product of any three of them is also of order two, they generate a group which involves an abelian subgroup composed of half the operators of the group, and each of the remaining operators is of order two and transforms each operator of this abelian subgroup into its inverse. Moreover, if this abelian group involves no more than $n-1$ invariants, the entire group of twice its order can always be generated by n such operators of order two. The necessary and sufficient conditions that the augmented right co-sets

$$H + HS_2 + HS_3 + \cdots + HS_\rho$$

constitute a group are that they include the product of any two of the operators $S_2, S_3, \cdots, S_\rho$ and that they also include the inverse of each operator in these co-sets. New abstract definitions for the general symmetric and alternating group of degree n were also given.

2. Dr. Bennett proves in this paper certain theorems concerning the transitive constituents of the maximal subgroup G_1 of a primitive group G. The primitive groups of degree 20 are then determined and it is shown that only four such groups exist. These are the symmetric and alternating groups of degree 20 and the two well-known Mathieu groups of degree $p+1$ and orders $p(p^2-1)$ and $\frac{1}{2}p(p^2-1)$.

3. In Professor Brenke's paper the following theorem is obtained concerning the series

$$a_1X_1 + a_2X_2 + \cdots + a_nX_n + \cdots,$$

where X_n denotes the Legendrian polynomial of degree n in x: The above series converges uniformly in any interval (c_1c_2) lying between -1 and $+1$ and not including either of those points, provided that
(1) it converges at one point of (c_1c_2);

(2) the series $\sum\limits_1^\infty A_nX_n$, where

$$A_n = \frac{n(n-1)}{2n-1}a_{n-1} - \frac{(n+1)(n+2)}{2n+3}a_{n+1},$$

converges uniformly in (c_1c_2);

(3) $$\lim_{n=\infty} a_n/\sqrt{n} = 0.$$

If the sum of the series be denoted by $f(x)$, this function will have in (c_1c_2) a continuous first derivative given by the equation

$$f'(x) = \frac{2a_1}{3(1-x^2)} + \frac{1}{x^2-1}\sum_1^\infty A_nX_n.$$

Example: Applied to the series

$$\frac{3}{2}X_1 + \frac{7}{3.4}X_3 + \frac{11}{5.6}X_5 + \cdots,$$

this gives

$$f'(x) = \frac{1}{1-x^2}, \quad \text{hence} \quad f(x) = \frac{1}{2}\log\frac{1+x}{1-x}.$$

4. In this paper Professor Frizell develops a procedure for producing sets of permutations of the elements of an ω series which exhibit all ordinal types of the second class.

The prime numbers form an ω series, the products of two, three, $\cdots$, n primes form series of types ω^2, ω^3, $\cdots$, ω^n respectively. Thus the set of all products of primes (i. e., the series of natural numbers) may be arranged in type ω^ω. Repetitions of this process combined with that of finite permutation explained in a previous paper yield permutations of the set of all natural numbers arranged in a series of type Ω.

The paper will appear in the *Science Bulletin* of the University of Kansas.

5. The General Analysis of Moore relates to certain properties of classes of real valued functions of a general variable. In this paper Professor Pitcher discusses certain classes of sequences (functions the range of whose variable is the denumerable class $1, 2, \cdots, n, \cdots$), showing which of the above mentioned properties are possessed by each class. The classes discussed are (a) all convergent series, (b) all convergent sequences, (c) all ultimately constant sequences. Each of these classes possesses properties which make the methods and theorems of general analysis applicable to it. The so-called $*$ extension of each of the above classes is computed.

6. Professor Davis considers the intersections with the cubic of a pencil of lines passing through a point upon the cubic. The imaginary intersections are represented by vectors along lines of the pencil joining points on a certain auxiliary cubic to points on a sextic complementary (in Poncelet's sense) to the original cubic. Through every point in the plane pass three such auxiliary cubics.

7. In Dr. Ingold's first paper use is made of the well known analogy between ordinary vectors and functions of a real variable x on an interval, say $a \leqq x \leqq b$; functions $f(x; s)$ are considered which involve a parameter s besides the variable x; the function f is regarded as the analogue of the defining vector of a curve in ordinary space. The extension of Frenet's formulas is used to define a set of functions analogous to the normal vectors of a curve. The usual vector formulas for curves in an n-dimensional space are then reproduced in terms of the function f and its derivatives with respect to the parameter s.

8. The second paper of Dr. Ingold contains a discussion of functions involving several parameters $u_1, u_2, \cdots, u_n$ besides the variable x on the interval $a \leqq x \leqq b$. A function

$f(x;\ u_1, u_2, \cdots, u_n)$ is regarded as the analogue of the defining vector of an n-dimensional surface. Functions analogous to normals to a surface are defined in terms of the first and second partial derivatives of f with respect to the parameters $u_1, \cdots, u_n$. The investigation is then limited to the case in which, for given values of the parameters u, the normals are all numerical multiples of the same function N and in which the derivatives of N with respect to the parameters are all linearly expressible in terms of the derivatives of f. With these restrictions on f the principal formulas of the differential geometry of n-dimensional surfaces in a $(n + 1)$-dimensional (euclidean) space, are obtained in terms of f and its derivatives.

9. Professor Cajori shows that Paolo Ruffini published in 1804 at Modena in Italy a monograph which describes a method of approximation to the real roots of numerical equations which is almost identical with the method given by W. G. Horner in 1819 in England. The difference is unimportant: the coefficients of the transformed equation appear, with Ruffini in a column, with Horner along a diagonal line, slanting upward from left to right.

10. In computing the southerly deviation of a falling body, one can not, with impunity, neglect the curvature of the lines of force in the neighborhood of the point P_0 (fixed with respect to the rotating earth) from which the body falls. Unless the lines of force of the rotating field are rectilinear (which, in particular, is the case when the field is constant), a plumb-line P_0R supported at P_0 does not coincide with the tangent t at P_0 to the line of force of the rotating field which passes through P_0. Some writers, Gauss for instance, measure the southerly deviation from the cone of revolution generated by t. In experiments, on the other hand, the southerly deviation is measured on the equipotential surface (of the rotating field) which passes through the plumb-bob R, from the circle of latitude generated by R. Therefore, in order to compare the results of experiment with those of theory, the latter method must be used. This method is the same as that described by Professor Roever in an earlier paper.*

In his present paper, a formula for the southerly devi-

* BULLETIN, vol. 16 (1910) p. 228.

ation (hereafter denoted by S. D.) is expressed in terms of the potential function $U_1 = f_1(r, z)$ (where r and z denote the distances of a general point from the axis of rotation OZ and the plane of the equator of the earth) of the earth's fixed field of force. The formula is of the form

$$\text{(1)} \qquad \text{S.D.} = Ah^2 + \cdots,$$

where $h = P_0R$ is the distance through which the body falls and A is a function of the earth's angular velocity of rotation ω and the first and second derivatives of f_1 for the point P_0, (r_0, z_0).

If, in particular, we assume the (rotating) field of force in the neighborhood of P_0 to be constant and of intensity g,

$$f_1 = f_2 - \tfrac{1}{2}\omega^2 r, \quad \text{where} \quad f_2 = -g \cos\phi \cdot r - g \sin\phi \cdot z,$$

in which the constant ϕ is the astronomical latitude at P_0. For this assumption formula (1) becomes

$$\text{(2)} \qquad \text{S.D.} = \tfrac{2}{3} \sin\phi \cos\phi \frac{\omega^2}{g} h^2,$$

or

$$\tfrac{1}{6} \sin\phi \cos\phi \, g\omega^2 t^4, \quad \text{since} \quad h = \tfrac{1}{2}gt^2 + \cdots$$

This is the formula given by Gauss* and later writers.

If, on the other hand,

$$f_1 = \frac{M}{\rho}\left[1 + \frac{\epsilon}{3}\frac{\rho_1^2}{\rho^2}\frac{r^2 - 2z^2}{\rho^2}\right], \dagger$$

where M is the mass of the earth, $\rho = \sqrt{r^2 + z^2}$, ρ_1 the mean radius of the earth and ϵ the constant .00167, formula (1) takes the form

$$\text{(3)} \qquad \text{S.D.} = .0055 \sin 2\phi_0 \cdot \frac{h^2}{\rho_0},$$

where ϕ_0 is the geocentric latitude of P_0 and ρ_0 is ρ for P_0.

* Gauss, Werke, vol. 5, 1867, p. 502.

† A formula given by H. Poincaré, Figures d'équilibre d'une masse fluide (1902), p. 107.

This formula gives values which are four and one half times as great as those of formula (2).

11. The purpose of Professor Young's note is to call attention to a very simple method of obtaining fundamental regions for cyclical groups of linear fractional transformations on two complex variables. The simplicity of the method is due to the fact that the determination of a fundamental region for a group of the specified kind is made to depend on the construction of a fundamental region for a simply isomorphic group on a single complex variable. The method, moreover, may be readily extended to the case in which the number of variables is n and to certain restricted types of groups which are not cyclical and not linear.

12. Continuing the work which appeared in the *Transactions* of January, 1908, Professor Kellogg makes an immediate extension to multiply connected regions. By means of the results Green's integral formula

$$u(x, y) = \frac{1}{2\pi} \int u(s) \frac{\partial G}{\partial n} ds$$

is rigorously established without any presuppositions concerning $u(x, y)$. From this criteria are obtained for the unique determination of harmonic functions by their boundary values in the case that these are not continuous.

13. At the last meeting of the Section, Professor Hedrick proved that a few important theorems of the theory of point sets hold true for any compact set of objects for which the first derived set is closed. In the present paper the hypothesis is added that any element of the set is enclosable in the interior of a family of diminishing assemblages in a uniform manner. With this hypothesis practically all of the fundamental theorems of point sets are shown to hold for general sets of objects; the list of theorems includes the general Heine-Borel theorem, the Cantor-Bendixon theorem, the Cauchy fundamental theorem, the oscillation theorem, and many others. No distance notion is used; those theorems which usually involve distance are reworded to avoid that concept.

O. D. KELLOGG,
Secretary of the Section.