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NOTE ON IMPLICIT FUNCTIONS DEFINED BY TWO EQUATIONS WHEN THE FUNCTIONAL DETERMINANT VANISHES.

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1. *Introduction.* Consider the two equations

$$(1)\quad f(x_1, x_2, \cdots, x_n; y, z) = 0, \quad g(x_1, x_2, \cdots, x_n; y, z) = 0,$$

and suppose a single point solution

$$(2)\qquad x_1 = a_1, x_2 = a_2, \cdots, x_n = a_n; \; y = b, z = c,$$

is known. Under certain well-known conditions, of which one is the non-vanishing of the functional determinant $\partial(f, g)/\partial(y, z)$ at the point in question, we may affirm that equations (1) define uniquely the functions

$$(3)\qquad y = \phi(x_1, x_2, \cdots, x_n), \quad z = \psi(x_1, x_2, \cdots, x_n)$$

in the neighborhood of the system of values (2). In general if the functional determinant vanishes, the functions (3) are multiple valued.* There are, however, certain exceptional cases in which the determination of y and z as functions $x_1, x_2, \cdots, x_n$ is unique although the functional determinant vanishes. It is proposed in this paper to examine briefly some of these exceptional cases.

In a certain trivial way the corresponding exceptional cases exist also when we consider a single equation defining one dependent variable. Suppose an equation for the determination of y as a function of x has the form

$$(4)\qquad f(x, y) \equiv x f_1(x, y) = 0.$$

* The case of analytic functions was treated by Professor G. A. Bliss in his Princeton Colloquium Lectures (1909). These lectures have not yet been published.

Evidently $f=0$ and $\partial f/\partial y=0$ for $x=y=0$, and the fundamental existence theorem is not applicable. But we may discard the factor x and if $f_1=0$, $\partial f_1/\partial y \neq 0$ for $x=y=0$, we may affirm the existence of a unique solution of the form $y=\phi(x)$. Geometrically the locus defined by equation (4) in the neighborhood of the origin consists of two branches; but one of these is the curve $x=0$, which is not expressible in the form $y=\phi(x)$. Again suppose the equation has the form

$$(5) \qquad f(x, y) = f_1^2(x, y) = 0,$$

where $f_1(0, 0)=0$. Here also $\partial f/\partial y=0$ at the origin; but we may consider the equation $f_1=0$ and, if $\partial f_1/\partial y \neq 0$ for $x=y=0$, we may affirm the existence of a unique solution of the form $y=\phi(x)$. Exceptional cases such as (4) and (5) for one equation are trivial because they are readily excluded by the usual assumption that f is irreducible in the arguments x and y.

In the case of two equations we shall assume always that each equation is irreducible; but this is not sufficient to exclude the exceptional cases which are similar, geometrically, to those presented by equations (4) and (5). The point of view is easily shown by some simple examples involving only one independent variable. Consider the two equations

$$(6) \qquad f \equiv y - x = 0, \quad g \equiv 2yz - x^2 = 0.$$

Both equations are satisfied for $x=y=z=0$ and $\partial(f, g)/\partial(y, z)=0$ at the origin. From the first equation $y=x$ and, when this value of y is substituted in the second, we get $x(2z-x)=0$. This equation is of the form (4) and has a unique solution $z=\frac{1}{2}x$. Hence for values of x different from zero the system (6) defines uniquely the functions

$$(7) \qquad y=\phi(x) \equiv x, \quad z=\psi(x) \equiv \tfrac{1}{2}x.$$

Geometrically the curve defined by (6) consists of two branches passing through the origin; but one branch is the z-axis and is not expressible by equations of the form (7). Again consider the two equations

$$(8) \; f \equiv z + 2y - 2x = 0, \; g \equiv 2z + 4y - 4x + 2yz - x^2 = 0.$$

Both equations are satisfied for $x=y=z=0$, and $\partial(f, g)/\partial(y, z)=0$ at the origin. From the first equation

$z = 2x - 2y$ and, when this value is substituted in the second, we get $(2y - x)^2 = 0$. This equation is of the form (5) and has a unique solution $y = \frac{1}{2}x$. Hence for values of x different from zero the system (8) defines uniquely the functions

$$(9) \qquad y = \phi(x) \equiv \tfrac{1}{2}x, \quad z = \psi(x) \equiv x.$$

Geometrically the curve defined by (8) consists of two coincident branches passing through the origin, that is, the two surfaces touch along the curve (9). Upon each surface the origin is an ordinary point.

The general method of investigating the solutions of equations (1), as illustrated by the preceding examples, is the following: Suppose one equation, say the first, can be solved by the fundamental theorem for one of the dependent variables, say y in terms of $z, x_1, \cdots, x_n$. When this value of y is substituted in the second equation it may happen that the latter becomes reducible in $z, x_1, \cdots, x_n$, and although the functional determinant vanishes the solution for z may be unique. For practical application it is convenient to formulate conditions upon f and g in order that this process shall lead to a unique result. Apparently no general formulation can be made, but it is possible to state theorems for special cases. Two of these theorems are given below.

2. In the theorem given in this section it is assumed that the functions f and g are real functions of real variables, and for convenience of reference the following statement of the fundamental theorem for two equations is given :

Fundamental Theorem. Consider the system of equations

$$(1) \quad f(x_1, x_2, \cdots, x_n;\, y, z) = 0, \quad g(x_1, x_2, \cdots, x_n;\, y, z) = 0,$$

and suppose a special solution

$$(2) \qquad x_1 = a_1, x_2 = a_2, \cdots, x_n = a_n;\; y = b, z = c,$$

is known. Suppose the functions f and g are continuous, and possess first partial derivatives which are continuous, in the neighborhood of the system of values (2). Suppose the functional determinant

$$\Delta = \begin{vmatrix} f_y & f_z \\ g_y & g_z \end{vmatrix}$$

does not vanish for the system of values (2).

Then there exists one and only one system of continuous functions of the form

$$A: \quad y = \phi(x_1, x_2, \cdots, x_n), \quad z = \psi(x_1, x_2, \cdots, x_n)$$

which satisfy equations (1) and the condition

$$b = \phi(a_1, a_2, \cdots, a_n), \quad c = \psi(a_1, a_2, \cdots, a_n).$$

Moreover the implicit functions A thus defined possess first partial derivatives which are continuous in the neighborhood of the system of values $x_1 = a_1$, $x_2 = a_2$, $\cdots$, $x_n = a_n$.

The solution described in the conclusion of this theorem will be referred to as a solution of the type A. When the functional determinant vanishes the following theorem states what seem to be the simplest conditions of practical value in order that there shall exist a unique solution of the type A. It is assumed in this theorem that f and g possess partial derivatives of the first and second* orders which are continuous in the neighborhood of the system of values (2).

THEOREM I. Consider equations (1) and suppose

$$(a) \qquad \begin{aligned} f(a_1, x_2, \cdots, x_n;\ b, z) &\equiv 0, \\ g(a_1, x_2, \cdots, x_n;\ b, z) &\equiv 0 \end{aligned} \qquad (\text{in } x_2, \cdots, x_n;\ z).$$

Suppose also

$$(b) \quad \Delta_1 = \begin{vmatrix} f_y & f_{x_1} \\ g_y & g_{x_1} \end{vmatrix} = 0, \qquad (c) \quad \Delta_2 = \begin{vmatrix} 0 & f_y & f_{x_1} \\ f_y & f_{yz} & f_{zx_1} \\ g_y & g_{yz} & g_{zx_1} \end{vmatrix} \neq 0$$

for the system of values (2).

Then there exists one and only one solution of the type A.

To sketch the proof of this theorem we observe that condition (c) implies either $f_y \neq 0$ or $g_y \neq 0$. Assuming $f_y \neq 0$ we may apply the fundamental theorem for one equation to the first of equations (1) and obtain a solution for y. From the first of conditions (a) it follows that when $x_1 = a_1$, $y = b$ *identically* in $x_2, \cdots, x_n, z$. Hence the solution for y has the form

$$(10) \qquad y - b = (x_1 - a_1)h(x_1, x_2, \cdots, x_n;\ z).$$

When this value of y is substituted in the second of equations (1) it follows from (a) that the equation assumes the form

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*It is not necessary to assume the existence and continuity of all the partial derivatives of the second order. No attempt is being made here to reduce these conditions to a minimum.

$$(11) \qquad (x_1 - a_1) g_1(x_1, x_2, \cdots, x_n; z) = 0.$$

Now the appropriate computation shows that [for the system of values (2)] condition (*b*) is equivalent to the condition $g_1 = 0$, and (*c*) is equivalent to the condition $\partial g_1 / \partial z \neq 0$. Hence we may discard the factor $x_1 - a_1$ in equation (11) and apply the fundamental theorem for one equation to obtain a solution $z = \psi(x_1, x_2, \cdots, x_n)$. When this value of z is substituted in (10) we have the final solution in the form

$$(12) \qquad y = \phi(x_1, x_2, \cdots, x_n), \quad z = \psi(x_1, x_2, \cdots, x_n).$$

Each step in the process yields a unique result; hence the solution (12) is unique and of type A.

3. An example * of a system of equations of the type to which Theorem I is applicable may be found in a problem in the calculus of variations. The problem has been treated † in a paper by G. A. Bliss in the BULLETIN, volume 13 (1907), page 321, and the system referred to is (3), page 322. The equations to be solved for s and t are

$$(13) \quad f \equiv \phi(s, t) - r \cos \alpha = 0, \quad g \equiv \psi(s, t) - r \sin \alpha = 0,$$

where the functions ϕ and ψ satisfy the conditions

$$(14) \qquad \begin{aligned} &\phi(0, t) \equiv 0, & &\psi(0, t) \equiv 0, \\ &\phi_s(0, t) \equiv \cos t, & &\psi_s(0, t) \equiv \sin t. \end{aligned}$$

Evidently a particular solution of equations (13) is

$$(15) \qquad r = 0, \quad \alpha = k, \quad s = 0, \quad t = k,$$

where k is any constant.

For the system of values (15) the functional determinant

$$\Delta = \begin{vmatrix} \phi_s & \phi_t \\ \psi_s & \psi_t \end{vmatrix} = \begin{vmatrix} \cos k & 0 \\ \sin k & 0 \end{vmatrix} = 0,$$

and the fundamental theorem is not applicable. We apply then the hypothesis of Theorem I. The identities (14) show

* Other examples are to be found in papers by F. R. Moulton, "A class of periodic solutions, etc.," *Transactions*, vol. 7 (1906), p. 546, equations (16); and by W. R. Longley, "A class of periodic orbits, etc.," *Transactions*, vol. 8 (1907), p. 166, equations (15).

† See also Bolza, Variationsrechnung, p. 270.

that condition (a) is satisfied. Also

$$\Delta_1 = \begin{vmatrix} f_s & f_r \\ g_s & g_r \end{vmatrix} = \begin{vmatrix} \cos k & -\cos k \\ \sin k & -\sin k \end{vmatrix} = 0,$$

$$\Delta_2 = \begin{vmatrix} 0 & f_s & f_r \\ f_s & f_{st} & f_{tr} \\ g_s & g_{st} & g_{tr} \end{vmatrix} = \begin{vmatrix} 0 & \cos k & -\cos k \\ \cos k & -\sin k & 0 \\ \sin k & \cos k & 0 \end{vmatrix} = -\cos k.$$

Hence if $\cos k \neq 0$ the hypothesis of Theorem I is satisfied and we may affirm the existence of a unique solution of the type A. If $\cos k = 0$, then $\sin k \neq 0$ and the apparent difficulty is overcome by interchanging the notation f and g in equations (13).

4. If the functions f and g in Theorem I are analytic, the process of solving the first of equations (1) for y, substituting in the second, and solving for z shows that the solution for y and z is also analytic. The solution may be obtained, however, without going through this process. For simplicity this will be shown when there is only one independent variable x, and we will suppose that the equations are satisfied for $x = y = z = 0$. If f and g are regular at the origin, equations (1) may be written

$$(16) \qquad f = \Sigma a_{ijk} x^i y^j z^k = 0, \quad g = \Sigma b_{ijk} x^i y^j z^k = 0.$$

For the system (16) the conditions (a), (b), and (c) are

$$(a) \qquad a_{00k} = 0, \quad b_{00k} = 0 \qquad (k = 1, 2, \cdots),$$

$$(b)\ \Delta_1 = \begin{vmatrix} a_{100} & a_{010} \\ b_{100} & b_{010} \end{vmatrix} = 0, \qquad (c)\ \Delta_2 = \begin{vmatrix} 0 & a_{010} & a_{100} \\ a_{010} & a_{011} & a_{101} \\ b_{010} & b_{011} & b_{101} \end{vmatrix} \neq 0.$$

The assumption (c) implies that either $a_{010} \neq 0$ or $b_{010} \neq 0$. Supposing a_{010} is not zero, we may without loss of generality take it equal to unity (by multiplying the series for f by $1/a_{010}$) and we may also take $\Delta_2 = 1$ (by multiplying the series for g by $1/\Delta_2$).

The solution to be determined has the form

$$(17) \qquad y = \alpha_1 x + \alpha_2 x^2 + \cdots, \quad z = \beta_1 x + \beta_2 x^2 + \cdots.$$

When the values of y and z from (17) are substituted in (16) the coefficient of each power of x must vanish. Equating to zero the coefficients of the first power of x we have

$$a_{100} + \alpha_1 = 0, \quad b_{100} + b_{010}\alpha_1 = 0.$$

From (b) it follows that these two equations are consistent and hence $\alpha_1 = -a_{100}$.

Equating to zero the coefficients of x^2, we have

$$(18) \quad \begin{aligned} &(a_{101} + a_{011}\alpha_1)\beta_1 + \alpha_2 + a_{200} + a_{020}\alpha_1^2 + a_{110}\alpha_1 = 0, \\ &(b_{101} + b_{011}\alpha_1)\beta_1 + b_{010}\alpha_2 + b_{200} + b_{020}\alpha_1^2 + b_{110}\alpha_1 = 0. \end{aligned}$$

When the value of α_1 is substituted in (18) it is found that the determinant of the coefficients of α_2 and β_1 is equal to $\Delta_2(=1)$. Hence equations (18) determine α_2 and β_1 uniquely as rational integral functions of the coefficients of (16) of order * less than 3.

Proceeding in this way, we may determine step by step the coefficients in the series (17). For suppose

$$\alpha_1, \cdots, \alpha_{n-1}, \beta_1, \cdots, \beta_{n-2}$$

have been determined as rational integral functions of the coefficients of (16) of order less than n. Equating to zero the coefficients of x^n we have

$$(a_{101} - a_{011}a_{100})\beta_{n-1} + \alpha_n + \phi_n = 0,$$

$$(b_{101} - b_{011}a_{100})\beta_{n-1} + b_{010}\alpha_n + \psi_n = 0,$$

where ϕ_n, ψ_n denote rational integral functions of a_{ijk}, b_{ijk} of order less than $n+1$. The determinant of the coefficients of β_{n-1} and α_n is equal to unity and hence these quantities are determined uniquely as rational integral functions of the coefficients of (16) of order less than $n+1$.

5. A method of extending the conditions of Theorem I is suggested by considering the geometric interpretation. Suppose the equations involve three variables x, y, z, which will be interpreted as rectangular coordinates in space. The two surfaces $f=0$, $g=0$ then define a curve and we suppose the hypothesis of Theorem I is satisfied at a point which will be taken for the origin. Then from the conditions of the theorem it follows that the curve consists of two branches passing through the origin; but one of these branches is the z-axis and is not expressible by equations of the form $y=\phi(x)$, $z=\psi(x)$.

*By the order of a coefficient is meant the number $i+j+k$.

It is apparent geometrically that if the curve consists of n branches passing through the origin, and if $n-1$ of these branches are plane curves lying in the yz-plane, while one branch does not lie in this plane, then the equations $f=0$, $g=0$ will have one and only one solution of the form $y=\phi(x)$, $z=\psi(x)$. When these geometric conditions are stated in analytic language we may formulate further theorems similar to Theorem I. Theorem II is given as an example.

THEOREM II. Consider equations (1) and suppose

$$(\alpha)\qquad \begin{aligned} f(a_1, x_2, \cdots, x_n;\ \lambda(z), z) &\equiv 0, \\ g(a_1, x_2, \cdots, x_n;\ \lambda(z), z) &\equiv 0, \end{aligned} \qquad (\text{in } x_2, \cdots, x_n;\ z)$$

where the function $\lambda(z)$, which is substituted for y, is continuous and has a continuous first derivative in the neighborhood of $z=c$, and $b=\lambda(c)$. Suppose also

$$(\beta)\qquad \Delta_1 = \begin{vmatrix} f_y & f_{x_1} \\ g_y & g_{x_1} \end{vmatrix} = 0,$$

$$(\gamma)\qquad \Delta_3 = \begin{vmatrix} 0 & f_y & f_{x_1} \\ f_y & f_{yz} + \lambda' f_{yy} & f_{zx_1} + \lambda' f_{yx_1} \\ g_y & g_{yz} + \lambda' g_{yy} & g_{zx_1} + \lambda' g_{yx_1} \end{vmatrix} \neq 0,$$

for the system of values (2). (The derivative of λ is denoted by λ'.)

Then there exists one and only one solution of the type A.

In order to prove this theorem we set

$$(19)\qquad y = \eta + \lambda(z)$$

in equations (1) and obtain the set

$$F(x_1, x_2, \cdots, x_n;\ \eta, z) \equiv f(x_1, x_2, \cdots, x_n;\ \eta + \lambda(z), z) = 0,$$

$$G(x_1, x_2, \cdots, x_n;\ \eta, z) \equiv g(x_1, x_2, \cdots, x_n;\ \eta + \lambda(z), z) = 0.$$

The condition (α) becomes

$$(a)\qquad \begin{aligned} F(a_1, x_2, \cdots, x_n;\ 0, z) &\equiv 0, \\ G(a_1, x_2, \cdots, x_n;\ 0, z) &\equiv 0, \end{aligned} \qquad (\text{in } x_2, \cdots, x_n;\ z)$$

and the conditions (β) and (γ) become, respectively,

$$(b)\qquad \begin{vmatrix} F_\eta & F_{x_1} \\ G_\eta & G_{x_1} \end{vmatrix} = 0, \qquad (c)\qquad \begin{vmatrix} 0 & F_\eta & F_{x_1} \\ F_\eta & F_{\eta z} & F_{zx_1} \\ G_\eta & G_{\eta z} & G_{zx_1} \end{vmatrix} \neq 0$$

for the system of values

$$x_1 = a_1, \quad x_2 = a_2, \cdots, x_n = a_n\,; \; \eta = 0, \quad z = c.$$

Hence the hypothesis of Theorem I is satisfied for the equations $F = 0$, $G = 0$, and we have a unique solution of the form

$$(20) \qquad \eta = \eta(x_1, x_2, \cdots, x_n), \quad z = \psi(x_1, x_2, \cdots, x_n),$$

such that

$$0 = \eta(a_1, a_2, \cdots, a_n), \quad c = \psi(a_1, a_2, \cdots, a_n).$$

Substituting the values of η and z in (19), we have the required solution for y.

If there is only one independent variable x, the conditions of Theorem II, for the system of values $x = y = z = 0$, imply that the equations have the form

$$f(x, y, z) \equiv [y - \lambda(z)] f_1(x, y, z) + x f_2(x, y, z) = 0,$$

$$g(x, y, z) \equiv [y - \lambda(z)] g_1(x, y, z) + x g_2(x, y, z) = 0.$$

The curve defined by these equations consists of two branches passing through the origin. One branch is the curve $y = \lambda(z)$ in the yz-plane and is not expressible by equations of the form $y = \phi(x)$, $z = \psi(x)$.

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STURM'S METHOD OF INTEGRATING $dx/\sqrt{X} + dy/\sqrt{Y} = 0$.

BY PROFESSOR F. H. SAFFORD.

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ONE of the simplest methods of obtaining the addition theorem for elliptic integrals of the first kind is based upon a method of integration which is usually referred to as Sturm's