NON-EUCLIDEAN GEOMETRY.

The Elements of Non-Euclidean Geometry. By JULIAN LOWELL COOLIDGE. Oxford, Clarendon Press, 1909. 8vo. 291 pp.

FOR some time there has been felt in our universities a lack of English texts in the branches of higher mathematics, while in the lower branches we have been literally flooded with them. If our books in the higher branches were to be merely good translations of the best that the German, French, and Italian have to offer, something is surely gained, but when the English texts bear all the ear marks of elegance of form, clearness and originality of presentation, and when they embody within them the spirit of research, they are not only worthy of the highest praise, but they should be received with open arms. It is therefore a great pleasure to note how mathematical literature in English has been enriched within the last year by two treatises in two such far-reaching and important subjects as the differential geometry of curves and surfaces * and non-euclidean geometry. Let us hope that the good work thus begun will continue until we shall have a mathematical literature of our own which will stand comparison with that of other nations.

The book under review, the first real treatise in non-euclidean geometry written in English, is at the same time a most noteworthy addition to mathematical literature in general. It is one of the books that come under the desirable category above described. It contains some original work, part of it hitherto unpublished. As far as the general makeup of the book is concerned, we should note the good table of contents and the excellent index. The axioms are printed in heavy type and the theorems are numbered, attention being called to these by italicizing the word *Theorem*, but not the body of the theorem. A large number of theorems are stated without proof — some because they follow easily from the preceding ones, and others because the author, as he expressly states, wishes to leave them as exercises for the reader. The discussion is somewhat too condensed in parts ; it would add a great deal to the value of the book if some portions were worked out

* A Treatise on the Differential Geometry of Curves and Surfaces. By L. P. Eisenhart, 1909.

in more detail — but this briefness is no doubt due to the large amount of ground covered in so little space. The reviewer would also like to see more words of explanation introduced in the proofs, and would appreciate a judicious use of paragraph markings and heavy type headings to indicate a passage from one set of related theorems to another set, thus relieving a monotonous appearance of the page and lending interest in the reading. There are copious foot-notes, giving detailed references and interesting comments on these references and on the text. There are several minor typographical errors but these offer no hindrance in the reading.

To understand clearly the author's purpose, let us quote from the preface. "Recent books dealing with non-euclidean geometry fall naturally into two classes. In the one we find the works of Killing, Liebmann, and Manning, who wish to build up certain clearly conceived geometrical systems and are careless of the details of the foundations on which all is to rest. In the other category are Hilbert, Vahlen, Veronese, and the authors of a goodly number of articles on the foundations of geometry. These writers deal at length with the consistency, significance, and logical independence of their assumptions, but do not go very far towards raising a superstructure on any one of the foundations suggested. The present work is, in a measure, an attempt to unite the two tendencies." The author has clearly and successfully done what he set himself to do. He has given a rigorous treatment of the foundations and having built strongly and wisely, he has raised a firm superstructure upon these. He has attacked the problem of non-euclidean geometry by its three approaches. (1) The elementary geometry of point, line, and distance (this development is the most complete and includes the first 17 chapters); (2) projective geometry and the theory of transformation groups (Chapter XVIII); (3) differential geometry, with the concept of distance element, extremal, and space constant (Chapter XIX). The only limitations as to subject matter are that the work does not go beyond three dimensions, thus gaining in clearness at the expense of generality, and that it treats only of the classical (i. e., archimedean and desarguian) systems.

Chapter I, "Foundations for a metrical geometry in a limited region," formulates the entities, definitions, and axioms upon which the geometry is founded. The author has found it best suited to his purpose to set up a system of axioms of his own, which is not however as condensed as some hitherto published. For the first fundamental undefinable, the *point* (as is usual) is taken, and for the second, from the mass of objects such as segment and motion, segment and order, motion, line and separation, and distance, following Peano and Levy, *distance* is chosen. Thus the existence of two classes of objects, points and distances, is posited. Then follow 17 axioms with their discussion and easily deducible theorems, and upon these the three-dimensional type of space is constructed. They include the positing of at least two points and a unique object called their distance, the usual congruent relations, the relations of greater than and less than, then Axiom XI, "If A and C be any two points, there exists such a point B distinct from either such that $\overline{AB} \equiv \overline{AC} + \overline{CB}$," which involves the existence of an infinite number of points and removes the possibility of a maximum distance; the axioms establishing a serial order among the points of a segment and its extensions, the extension beyond the geometry of a single line and that beyond the geometry of a single plane.

Chapter II deals with "Congruent transformations." Having taken *distance* as undefinable and distance being a magnitude, the basis for a metrical geometry has been laid. To complete the metrical system, two more assumptions are added. The first is the axiom of continuity — this leads to several theorems, the archimedian axiom, the introduction of the concept of number as the numerical measure of two distances, and its extension to the irrational number. Having defined a congruent transformation between two sets of points (P) and (Q), it is by means of a second axiom which allows the enlarging of a congruent transformation to include additional points, and upon the definition of angle and its numerical measure, that all the properties of congruent figures and the relation of perpendicularity in the plane and in space are developed.

In the last chapter, dealing with the comparison of distances and angles and their numerical measures, the question of the sum of the angles of a triangle was not considered. It is the purpose of Chapter III, "The three hypotheses," to fill in this gap. Here we meet the theorems concerning the continuous change of distances and angles. Thus: "If in any triangle, one side and an adjacent angle remain fixed, while the other side, including this angle, may be diminished at will, then the external angle opposite to the fixed side will take and retain a value differing from that of the fixed angle by less than any assigned value." It follows that in such a triangle the sum of the angles can be made to differ infinitesimally from a straight angle. Then follows the definition of quadrilateral. Then the theorems: If there exist a single rectangle, every isosceles birectangular quadrilateral is a rectangle; if there exist a single right triangle (any triangle) the sum of whose angles is congruent to, less than, or greater than a straight angle, the same is true of every right triangle (any triangle). According to the assumptions, then, of the existence of a single triangle the sum of whose angles is congruent to, less than, greater than a straight angle, we have the parabolic (euclidean), hyperbolic (lobachevskian), elliptic (riemannian) geometries. The chapter closes by showing that the euclidean hypothesis holds in the infinitesimal domain. The development is complete throughout and very satisfactory.

Having now completed the foundations in one direction, the materials out of which the superstructure is to be erected are gotten ready in this and the succeeding chapters. The development of the trigonometric formulas (Chapter IV) depends upon the proofs of the existence of two limits and of the continuity of two functions, of a distance and of an angle respectively. These are (1) the limit of the fraction $\text{M}\overline{CD}/\text{M}\overline{AB}$ ($\text{M}\overline{CD}$ means the measure of $\overline{CD}$ in terms of some convenient unit) in an isosceles birectangular triangle $ABCD$ whose right angles are at A and B, where $\overline{AB}$ becomes infinitesimally small while $\overline{AD}$ remains constant; and the continuity of the resulting function of $\text{M}\overline{AD}$; this function $\phi(x)$ is shown to have the property $\phi(x+y)+\phi(x-y)=2\phi(x)\phi(y)$, whose solution is easily seen to be

$$\phi(x)=\cos\frac{x}{k}=1-\frac{x^2}{k^2\cdot 2\,!}+\frac{x^4}{k^4\cdot 4\,!}\cdots.$$

This constant k is later shown to be the radius of a euclidean sphere upon which the non-euclidean plane may be developed, and therefore $1/k^2$ is called the *measure of curvature* of space. This again leads to the distinction between the three geometries, $1/k^2 \lesseqqgtr 0$, according as we have the hyperbolic, parabolic, or elliptic case. (2) The limit of the fraction $\overline{AB}/\overline{AC}$ in the right triangle ABC, right angled at B, where AB becomes infinitesimal while $\angle BAC$ is constant. The resulting function

$f(\theta)$ of $\angle BAC$ is proved to be continuous, and it obeys the law $f(\theta + \phi) + f(\theta - \phi) = 2f(\theta)f(\phi)$, the same law as $\phi(x)$ above does. Hence $f(\theta) = \cos \theta / l = \cos \theta$ if l is so chosen that the measure of right angle is $\pi/2$. It also follows that $\lim \overline{BC}/\overline{AC} = \sin \theta$. The development of the usual trigonometric formulas (corresponding to the spherical trigonometric formulas upon a sphere of radius k) now follows, and each is shown to be universally true, and reduces to that of the euclidean plane if we put $1/k^2 = 0$. The discussion is shortened by the use of the consideration that we have euclidean geometry in the infinitesimal domain. The entire chapter is a bit of rigorous work to be highly commended. Chapter V, "Analytic formulæ," opens by introducing the idea of directed distances and angles and their measurement in the plane and in space. The usual coordinate systems are set up, followed by the passage to homogeneous coordinates, the formulas for the distance between two points and the distance from a point to a plane. The element of arc length is computed and by comparison with the usual distance formula, it is shown that the non-euclidean plane may be developed upon a surface of constant curvature $1/k^2$ in euclidean space. The theorem is proven that every congruent transformation of space is represented by an orthogonal substitution in the homogeneous variables $x_0 : x_1 : x_2 : x_3$.

To complete the foundations and render them one homogeneous whole, and do away with a disadvantage under which he has been laboring from the start, the author now sets himself the task of showing (1) that the assumptions made at the outset are consistent, and (2) what degree of precision might be given to Axiom XI, where it was assumed that any segment might be extended beyond either extremity. These points are respectively dealt with in Chapter VI, "Consistency a significance of the axioms," and Chapter VII, "The geometric and analytic extension of space." That the axioms are sufficient has been shown by the possibility of expressing distances and angles analytically. The axioms are shown to be compatible by setting up actual systems of objects in the plane and in space obeying them and the three hypotheses. Their mutual independence is finally examined in the usual way. The author now passes to the extension of Axiom XI. It is easily shown that under the parabolic and hyperbolic hypotheses, any segment may be extended beyond either extremity by any desired amount. This is not so under the elliptic hypothesis. To get the desired result, the author sets up six axioms, I′–VI′, of which the first two are: there exists a class of objects containing at least two members, called points; and every point belongs to a subclass (called a consistent region) obeying Axioms I–XIX. The others state properties of such consistent regions, e. g., any two consistent regions, having a common point, have a common consistent region. By means of overlapping consistent regions, we may by a process of analytic extension, reach a set of coordinates for every point in space. A point has a unique set of coordinates. But while under the parabolic and hyperbolic hypothesis there is but one point for each set of coordinates, under the elliptic hypothesis there are two possibilities: (1) there is but one point for each set of coordinates (elliptic space); (2) there are two points (called equivalent points) at a distance $k\pi$ which have the same set of coordinates (spherical space). From the additional axioms it also follows that there must exist, under the elliptic hypothesis, a point having any chosen set of homogeneous coördinates not all zero. This is not so in the other two cases, and the author brings these two up to an equality with the first by the extension of the concept point so as to include *ideal* elements. It is also shown how to find figures corresponding to imaginary coordinate values. The extension is completed by extending the concepts distance and angle to fit the extended space. The remainder of the chapter shows how this extension may be made by the use of the *absolute* and the *cross-ratio*, in the highly interesting procedure of Cayley's Sixth memoir upon quantics.

Chapter VIII is a discussion of "The groups of congruent transformations." Having defined a congruent transformation as any collineation of non-euclidean space that keeps the absolute invariant, and having shown that every such collineation is a congruent transformation, the author now sets up the groups and subgroups of congruent transformations, by studying the collineations of space that carry the absolute into itself, and viewing the matter both from an analytic and geometric standpoint.

Chapter IX deals with "Point, line, and plane treated analytically." Having now completed the foundations and gotten ready the machinery for treating space as a perfect analytic continuum, the author launches forth to build up his superstructure. And having built strongly and surely, the task is not a difficult one.

The object of this chapter is to express the fundamental metrical properties of the point, line, and plane in terms of the invariants of the congruent group. The absolute is the basis of all the work. The theorems are proven, using the nomenclature of the point geometry, but they are stated, in their full duality, in parallel columns. The work moves along rapidly. It includes theorems on the centers of gravity of two or more points, loci of second order, the parallel angle, the desmic configuration, and paratactic lines.

This last chapter included an introduction of the non-euclidean line geometry, using the line as element. Chapter X, "The higher line geometry," is a highly interesting continuation of that work, using a pair of lines, invariantly connected, as element. This element is the *cross* — the proper cross is a pair of real lines mutually absolute polar, neither of which is tangent to the absolute, and determining a pencil of coaxial complexes. The Plücker coordinates of the pencil are used as the coordinates of the cross, which consists of the directrices of the common congruence. In the case where one pair of lines is tangent to the absolute, all the complexes of the pencil are special, and they may be determined by any pair of a pencil of tangents; this pencil of tangents is called an improper cross. The geometry of the cross in hyperbolic space throws light upon the geometry of the point in the complex elliptic plane, for there exists a one-to-one correspondence between the two assemblages. The geometry of the cross in elliptic space, on the other hand, finds its counterpart in the geometry of pairs of points, one in each of two real planes. The simplest configurations of crosses, such as the chain and synectic congruence, are then studied and carried over to their corresponding analogues mentioned above. The Clifford surface is briefly touched here. The development of the cross-geometry in elliptic space is the author's own.

The following three chapters: Chapter XI, "The circle and the sphere," Chapter XII, "Conic sections," and Chapter XIII, "Quadric surfaces," are analytic studies, including complete classifications, of the metrical properties of these concepts and their configurations in non-euclidean space from the dual standpoint as loci and as envelopes. The first of these chapters ends with an elegant transformation from euclidean to non-euclidean space, showing that the Darboux-Dupin theorem (in any triply orthogonal system of surfaces, the intersection lines are lines of curvature) must hold in hyperbolic space; the last one terminates with the introduction of elliptic coordinates, and a pretty bit of work whereby Staude's ring construction of the ellipsoid is extended to non-euclidean space.

Chapter XIV discusses "Areas and volumes." The first function developed is the *sine amplitude* of a triangle in terms of the sides and angles, which is closely analogous to the corresponding area of a euclidean triangle; indeed if we put $1/k^2 = 0$, then $k^2 \sin(ABC) = 2$ area $\triangle ABC$. The area of a bounded surface is now defined to be the limit of a sum of infinitesimal quadrilaterals (triangles) covering the surface; this corresponds to the definite integral, and the area of a triangle is shown to be equal to the quotient of the excess $(A + B + C - \pi)$ by the measure of curvature of space. The sine amplitude also appears in considering volumes of tetrahedrons; again if $1/k^2 = 0$, then $k^3 \sin(ABCD) = 6$ volume tetrahedron $ABCD$. Following Schläfli, a formula for the volume of a tetrahedron as a definite integral is set up. The volumes of a cone of revolution, of a sphere, and the total volume of elliptic and spherical space, are also found.

In Chapters XV, "Introduction to differential geometry," and XVI, "Differential line geometry," the differential geometry of non-euclidean space is dealt with. The developments in the first of these follow the general scheme worked out for the euclidean case in Bianchi,* but the method is different from the latter's development of the non-euclidean case.† Starting with the striking theorem: "the square of the curvature of a curve is the square of its curvature treated as a curve in a four-dimensional euclidean space, minus the measure of curvature of the non-euclidean space," and finishing with the theorems on minimal surfaces, the reviewer has found this chapter as well as the succeeding one, some of the pleasantest reading and most interesting in the entire book. In the latter chapter, the line geometry as developed in Chapters X and XV receives its final treatment. It is particularly a study of the line congruence—the general properties concerning the limiting and focal points and planes, the normal and isotropic congruences. The theorems are stated in all their duality, including some of the author's own work on isotropic congruences. The chapter ends by

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* Bianchi-Lukat, Vorlesungen über Differentialgeometrie, chapters I, III, IV, VI.

† Ibid., chapters XXI, XXII.

pointing out the remarkable correspondence existing between the theory of rays in hyperbolic and elliptic space.

In Chapter XVII, "Multiply connected spaces," the author returns to complete the problem of Chapter VII. There, the proof that to each point there corresponded but a single set of homogeneous coordinates depended upon Axiom VI′, which required that "a congruent transformation of any consistent region may be enlarged in a single way to be a congruent transformation of every point." Casting this one assumption aside and keeping all the others, is it possible to have a space where each point shall correspond to several sets of coordinate values? Such spaces are found obeying Axioms I′–V′, and in the sense of analysis situs these are multiply connected. As for the congruent transformations which lead to the group of identical transformations of a multiply connected space, examples are found for the euclidean plane and space, and some little known examples for the hyperbolic case; there are no multiply connected elliptic planes, but there are such three-dimensional spaces.

Chapter XVIII, "The projective basis of non-euclidean geometry," tends to base the metrical non-euclidean geometry upon projective considerations — and truly so, for the former depended upon the cross-ratio which is a projective concept. The author thus starts anew, sets up a system of eleven axioms for projective geometry, taking *point*, *line*, and *separation* as fundamentals, points out the invariance of the cross-ratio, sets up the coordinate system for the point in the line and plane by means of this, also the equations of line, plane, and quadric. There are now added five axioms regarding the laws obeyed by an assemblage of transformations called congruent transformations; and a quadric cone (absolute) is found which is invariant under these; two other axioms now serve to define distance by means of a cross-ratio and the previous distance formula is finally reached; hence we have the conclusion that this set of eighteen axioms are compatible with the hyperbolic, elliptic, or parabolic hypothesis, and with these only.

In the concluding Chapter XIX, "The differential basis for euclidean and non-euclidean geometry," the problem is finally attacked from the differential viewpoint. It was shown that a non-euclidean plane was a surface of Gaussian curvature $1/k^2$; further that the sum of the distances from a point to any other two, not collinear with it, was greater than the distance of these latter; thus the straight line ought to be looked upon as a geodesic, and a plane may be generated by a pencil of geodesics through a point. The problem then is: to determine the nature of a three-dimensional point manifold which possesses the property that every surface generated by a pencil of geodesics has constant Gaussian curvature. The concepts *point* and *correspondence of point and coordinate set* are taken as fundamental. Three axioms are set up — one positing a restricted region, the second setting up the differential distance formula in such a region, and the third positing congruent transformations between arcs of geodesics. By the introduction of curvature of space, the previous distance formulas are again arrived at, and we draw the conclusion that these three axioms are compatible with the euclidean, hyperbolic, and elliptic hypothesis, and with these alone.

The chapter terminates with a brief summary of the entire work, in which the author discusses the pros and cons of the three methods of attack of the problem of non-euclidean geometry, viz., the first method developed in the first seventeen chapters, the projective method of Chapter XVIII, and the differential method of Chapter XIX. The last of these is no doubt the quickest and most direct, but this directness has been gained at the high price of "assuming at the outset that space is something whose elements depend in a definite manner on three independent parameters," instead of the more abstract view, which looks upon space as a set of objects which can be arranged in multiple series. The author concludes, and we join with him in this, that there is no answer to the question which method is the best, but that the choice is a matter of personal æsthetic preference.

In conclusion, the reviewer might again call attention to the elegant and vigorous style, to the very thorough and consistent treatment of the subject, and to the logical building up of the material step by step into one complete and harmonious whole. We might close with the author's words: "Let us not forget that, in a large measure, we study pure mathematics to satisfy an æsthetic need."

JOSEPH LIPKE.