itself to all readers, namely, a separate caption for each page intended to indicate briefly the contents of that page. This is especially useful as the author seldom sums up results in a way readily to catch the reader's attention.

J. I. HUTCHINSON.

BÔCHER'S HIGHER ALGEBRA.

Introduction to Higher Algebra. By MAXIME BÔCHER, Professor of Mathematics in Harvard University; prepared for publication with the cooperation of E. P. R. DUVAL, Instructor in Mathematics in the University of Wisconsin. New York, Macmillan, 1907. xi + 321 pp.

Einführung in die höhere Algebra. Von MAXIME BÔCHER, Deutsch von HANS BECK, mit einem Geleitwort von E. STUDY. Leipzig, Teubner, 1910. xii + 348 pp.

THE term "higher algebra" has been so often used in America to denote a very low type of merely formal algebra and to include subjects like infinite series, which are not properly algebraic at all, that it is refreshing to find a book like this one of Professor Bôcher's, which really corresponds to its title. It does so, not only by reason of the purely algebraic character of its material, but also because this material is worked up in a strictly logical as well as systematic manner.

The amount of available algebraic material is so enormous, and it branches out in so many different directions, that some selection is inevitable; even the extensive two-volume works of Weber and Netto are confined to certain special lines. The volume under review aims to furnish the reader with an *introduction* to the whole field, to lay a broad and deep foundation for further study, and in particular, to give an adequate algebraic preparation for the study of modern analytic geometry. This aim has been accomplished with remarkable success.

There is one special topic, however, to which the author gives more than an introduction, and that is the theory of elementary divisors (Elementarteiler). In the last three chapters he not only introduces elementary divisors in a most expeditious and satisfactory manner, but carries their theory through to a fair degree of completeness, so far as the more important applications are concerned.

It has for a long time seemed to me that the theory of elementary divisors was destined to assume a much more prominent position in the science of mathematics than has hitherto been given it. For in all the applications of linear substitutions and quadratic forms, whether to geometry, algebra, or the theory of numbers, there is a whole class of problems whose complete solution is essentially dependent on elementary divisors. This belief is strengthened by observing that the subject has recently found a place in several text-books, including Muth's Elementarteiler, Bertini's Geometria Proiettiva degli Iperspazi, Bromwich's Quadratic Forms, Kowalewski's Determinantentheorie, as well as Bôcher's Algebra.

In the early chapters the author, after deriving some of the elementary properties of polynomials and determinants, considers the theory of linear dependence and the solution of systems of linear equations, as based on the idea of the rank of a matrix. The great progress that has been made in recent years in the theory of linear equations is vividly illustrated by the striking contrast between the simplicity and completeness of Bôcher's treatment in Chapter IV and the complexity and incompleteness of Chrystal's treatment in the sixteenth chapter of the first volume of his Algebra, published in 1886.

The author then defines a matrix, namely, a square array of n^2 ordinary numbers, as a single complex quantity (hypercomplex number), and develops the algebra of matrices in such a way as to be able to apply it to linear transformations, collineations, bilinear forms, and quadratic forms. He very wisely makes these subjects concrete and tangible by constantly keeping their geometric significance before the reader.

The subject of invariants is considered from a sufficiently broad standpoint to be applicable not merely to the classical theory of the invariants of n-ary forms, but to all the mathematical theories in which invariants occur; and surely that includes a very extensive category. The reduction of a quadratic form to a sum of squares, the law of inertia for real quadratic forms, the properties of a system of a quadratic form and one or more linear forms, and the simultaneous reduction, in two special cases, of a pair of quadratic forms to sums of squares, are taken up in order. The general problem of the simultaneous reduction of a pair of quadratic forms to a normal form is postponed to the last chapter, where it becomes solvable by means of elementary divisors.

The precise nature of the reducibility of a polynomial in an arbitrary domain of rationality and of the greatest common divisor of two polynomials is carefully explained, so as to afford a firm basis for the treatment of elimination, and of resultants and discriminants. Finally, after the theory of elementary divisors has been developed, it is applied to the important problems of the classification of collineations and the classification of pairs of quadratic forms.

It is to be noticed that Galois's theory of equations and the theory of permutation groups, which necessarily accompanies it, are not included in the scope of the work, although the group concept is introduced in connection with linear transformations.

A very unusual feature is the way in which the material is systematized and unified; the connections between different lines of thought are pointed out; the origin, significance, and application of every new idea are carefully indicated. This makes the work an ideal text-book, as I have found by actual trial in the class room.

Naturally, no two persons would quite agree in their choice of the tools to be employed. Personally, it seems to me that modular systems might well have been utilized at certain points, somewhat as in Pund's algebra.

Throughout the literature of linear transformations there is a very common confusion arising from the failure to distinguish in language between the two transformations $x_i = \Sigma a_{ij} x'_j$ and $x'_i = \Sigma a_{ij} x_j$, either being referred to indiscriminately as the linear transformation of matrix a. In Bôcher's book the context usually indicates which is meant, but clearness would be gained by a more explicit statement.

There are a few trifling errors and misprints, but they will cause the reader no inconvenience, and most of them have been eliminated in the German translation. There is a good index in the original and a still better one in the German edition.

Although in a rapidly growing science like mathematics the best possible text-book must necessarily be restricted to a brief period of usefulness, yet it seems evident that this one will remain a classic for a considerable time to come.

The very existence of a German edition is a distinct compliment, not only to Professor Bôcher, but to American scholarship as well. If we are to judge by the giant strides that mathematical science is now making in this country, similar compliments will become more frequent in the future.

ARTHUR RANUM.