structive, or, at least, discriminating. The need of improvement in line with the constructive criticism is, in the judgment of the present reviewer, freely recognized and continually better met.

H. W. TYLER.

OSGOOD'S CALCULUS.

A First Course in the Differential and Integral Calculus. By WILLIAM F. OSGOOD, PH.D., Professor of Mathematics in Harvard University. New York, the Macmillan Company, 1907, pp. xv + 423. Revised edition, 1909, pp. xv + 462.

PROFESSOR OSGOOD in his presidential address before the AMERICAN MATHEMATICAL SOCIETY * has discussed and illustrated the principles which his experience has led him to consider should govern the teaching of the calculus. In the present text he gives us the detailed application of those principles to the difficult pedagogical problems which confront the instructor in the first course in this subject.

Successful instruction in mathematics requires careful adjustment of the conflicting claims of rigor, formalism, and interest. Professor Osgood has recognized † that rigor is a relative matter particularly in elementary instruction, and has enunciated the principle that in such instruction a discussion is to be regarded as rigorous if it meets all the logical demands which the student can be regarded as capable of appreciating at that time. This principle is at bottom the same as that which governs contemporary judgment of productive work, and its application to instruction is but a recognition of the fact that the mathematical development of the individual differs in general from that of the race at most by a transformation of similarity.‡ It is evident, however, that such a principle must be applied with care, for otherwise it may be cited in defense of a multitude of mathematical sins. If used with judgment, however, as is the case in this text, it becomes the very foundation of successful mathematical teaching.

The applications of this principle are in evidence throughout the book. For example: the theorem on the limit of the sum is at first (page 12) tacitly assumed, then (page 14) mentioned in a footnote, and finally proved (page 15) when the progress

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* BULLETIN, vol. 13 (1907), pp. 449–467.

† *Annals of Mathematics*, ser. 2, vol. 4 (1903), p. 178.

‡ Cf. Cantor, Geschichte der Mathematik, 3 Aufl., Bd. 1, p. 3.

of the work has rendered evident its necessity. Again in the discussion of $D_x(u + v) = D_x u + D_x v$ and similar theorems (pages 14, 21) the existence of $D_x u$ and $D_x v$ is assumed without comment, but later (page 423) the assumption and its significance are discussed. A similar point is illustrated in the treatment of $D_x y = D_t y / D_t x$ (pages 23, 423).

It is in conformity with this principle too, that Professor Osgood, an authority on questions of uniform convergence, has omitted this important subject from his text. The omission is at first sight somewhat startling in view of the author's tendencies, and yet when one reflects that the genius of a Weierstrass was required to establish uniform convergence as a constructive principle, the wisdom of omitting the subject from a course for beginners is apparent. As a consequence of this omission the continuity, differentiation, and integration of series are merely mentioned with a reference to the author's Infinite Series.

Progressive criticism of results already obtained and of methods already used must form an important part of a system of instruction which continually modifies its standard of rigor to accord with the student's increasing power of comprehension. This criticism is the more valuable to the student in proportion as it is constructive rather than destructive. It is therefore desirable that the necessarily incomplete discussions of an elementary course should be susceptible of ready modification in order to satisfy later more exacting demands, but they should not be such as to require extensive reconstruction. A case in point here is the treatment of Duhamel's theorem (page 164). The theorem is given in the original inexact form in which the question of uniformity is so veiled that in order to bring out the precise point at issue considerable destructive criticism of the type which Professor Osgood himself has given * is rendered necessary. Professor Osgood's own revised formulation † is as simple in statement and proof as the original, if the question of uniformity is waived, and it has the important advantage of needing only to be supplemented rather than reconstructed in a later and more advanced course. The revised form has, moreover, the further advantage of more direct application to the problem wherein the theorem is most valuable, namely, that of the formulation of questions involving the definite integral as the limit of a sum.

* Osgood, *Annals of Mathematics*, ser. 2, vol. 5 (1903), p. 169ff.
† Ibid., p. 173

The formal and manipulative side of the calculus has till recently received principal emphasis in American texts, and indeed to such an extent that the student has too frequently failed to obtain an adequate conception of the fundamental ideas of the subject and their application to concrete problems.

On these fundamental ideas and methods rather than on mere manipulation the stress is laid in Professor Osgood's text. For example an extended graphical and analytical treatment of $D_x x^n = nx^{n-1}$ (n fractional and negative) is given; the existence of $e = \lim_{n=\infty} (1 + 1/n)^n$ is accurately though simply established, and the idea of the definite integral as the limit of a sum is developed with careful attention to detail by means of a series of geometrical and statical problems. On the other hand in place of the usual treatment of the integration of rational fractions, of $\int R(x, \sqrt{ax^2 + bx + c})dx$, and of $\int x^m(a + bx^n)^p dx$, there is a short introduction to the use of the well known integral tables of B. O. Peirce. The most serious objection to this departure from the conventional treatment lies in the fact that indefinite integration, as an inverse operation, is always a difficult process for the student, and the power to reduce a given function to a known integrable form, *e. g.*, to one of those in the tables, is in general gained only by considerable directed practice. The traditional discussion of the types above mentioned may well be displaced by an introduction to the use of the tables, but in the opinion of the reviewer that introduction should contain a somewhat detailed discussion of the simpler integrable forms; a statement concerning forms not in general integrable, *e. g.*, $\int R(x, \sqrt{ax^3 + 3bx^2 + 3cx + d})dx$; and a statement at least of the theorems concerning the integrability of $R(x)$, $R(x, \sqrt{ax^2 + bx + c})$, $R(\sin x, \cos x)$, etc.

Abundant opportunity for practice and differentiation and integration is afforded by the supplementary exercises (revised edition, pages 324–359).

The space and time saved in the chapter on integration has been used to most excellent advantage in that on partial differentiation.

In much of the work of the elementary course in the calculus the analytical difficulties are those arising in the proofs of the theorems used. Here the trouble is almost invariably one of interpretation. The meaning of the partial derivative especially when several dependent and independent variables occur is seldom understood. The important section (page 306) on the notation used for partial derivatives, and the problems 4, 8, 21, 24, pages 312–315 form a much needed addition to the usual treatment of this subject, and will be a welcome aid to the teacher and student of thermodynamics. Were such discussions more common in texts on the calculus, their proper place, they would be unnecessary in treatises on thermodynamics.* The treatment of the various cases of $du = \partial u/\partial x\, dx + \partial u/\partial y\, dy$, etc. (page 292ff.), is another valuable aid to the clarification of the student's ideas concerning partial differentiation, and together with the treatment of the differential of a function of a single variable (page 92ff.) applies the principles concerning differentials enunciated in the author's presidential address † in such a way as to leave no opportunity for the "little zeros" to insinuate themselves into the student's mind.

Professor Osgood has long emphasized by precept and example the importance of developing new mathematical concepts in the student's mind by means of problems, *i. e.*, of causing the new mathematical idea to appear as a necessary element for the solution of a definite geometrical or physical problem.

His chapter on the definite integral as the limit of the sum is an example of this method, as well as of gradual development of precision in concept and demonstration conformably to the student's advance in assimilative power. The important purely theoretical matters in such a chapter are the concept of the definite integral as the limit of a sum, the upper and lower integrals, the fundamental theorem of the integral calculus, the proof of the existence of the definite integral, and the theorem of Duhamel. In Professor Osgood's treatment the area problem leads (page 153) to the concept of the definite integral as the analytic formulation of a limit the existence of which is geometrically evident. A previous discussion (pages 111–2) had developed the relation $d(\text{area})/dx = \text{ordinate}$. Comparison gives the fundamental theorem of the integral calculus. Of course the existence of an area is tacitly assumed, but that is at this stage of minor importance. The essential thing is that the student visualize the definite integral and understand its relation to the anti-derivative, *i. e.*, that he recognize the first as formulating his problem, the second as a means of evaluating its result, and shall see the necessity of the fundamental theorem

* E. g., Bryan, Thermodynamics, Chapter III, pp. 21–26.

† BULLETIN, loc. cit., p. 451.

of the integral calculus as the link which connects his method of formulation with his method of evaluation.

The next step is the introduction of the upper and lower integrals.* The problem of the fluid pressure upon a vertical plane surface with curvilinear boundary leads to the establishment of the element of the upper and lower integrals, their relation to the element of the integral and the passage to the limit which shows the existence of the upper and lower integrals, their equality and the existence of the integral. Here the reasoning depends on geometrical and statical considerations.

The way is now prepared for the analytic proof of the existence theorem for the definite integral. The proof is given in the usual way for a monotonic function. Though it is illustrated by the figure of the circumscribed and inscribed areas, and the language is in part geometrical, no fact is used the arithmetic truth of which is not immediately evident from the definition of the monotonic function. The extension to the case of a finite number of maxima and minima is then made, and finally in the appendix (page 423) attention is called to the case of functions with an infinite number of maxima and minima.

The chapter is noteworthy for the manner in which the difficulties of the theory of the definite integral are brought to the student's attention as inherent in the problems to which the integral calculus is applicable.

The chapters on double and triple integrals follow the same lines and there is the same gradual development of the fundamental concepts and avoidance of unnecessary analytical difficulties. The insistence on the distinction between the double (or triple) integral and the iterated integral, and on the fundamental theorem of the integral calculus as the link which joins them, is as desirable as it is unusual in elementary texts.

Few students, even those for whom mathematics is an elective rather than a required subject find their chief interest in the precision of its concepts, the rigor of its demonstrations or the elegance of its formulas. They continually demand immediate application to concrete problems as a reason for considering the subject at all.

Professor Osgood's treatment is well fitted to satisfy this demand. For example, as soon as the derivative has been defined

* The terms upper and lower integrals are not mentioned, but the concept is used.

and a few rules of differentiation established, the new method is applied to problems of maxima and minima, of velocity, of curve tracing, and of roots of equations in such a way as to make evident its power as compared with that of elementary methods. The result is that the first three chapters would by themselves form an excellent introduction to the differential calculus for those students who desire a knowledge of its scope and methods without going into analytical details.

The earlier American texts in the calculus have in general confined their applications to the field of geometry. The adherents of the "Perry movement" have gone to the other extreme and have produced a flood of problems taken from engineering practice. They have, however, too often forgotten that the student for whom the problems are designed has not yet acquired the technical knowledge necessary for an appreciation of their meaning and importance.

Professor Osgood has by no means neglected geometry, as his chapters on the cycloid, on curvature and evolutes, on envelopes, on partial differentiation, and on definite integrals bear witness. On the other hand, he has throughout the text, and especially in the chapter on mechanics, provided numerous problems illustrating the application of the calculus to physical phenomena, and that without requiring more technical knowledge than can be introduced into the text without distracting the student's attention from the true aim of the work — the application of the calculus to physical phenomena, as opposed to the formulation of a particular physical problem.

In conclusion it may be said that Professor Osgood's text is characterized by insistence on the real essentials of the calculus, and by consistent maintenance of that close relation between theory and application to which both pure and applied mathematics owe their most important advances.

CHARLES N. HASKINS.

THE UNIVERSITY OF ILLINOIS,
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