

THE FIFTEENTH ANNUAL MEETING OF THE
AMERICAN MATHEMATICAL SOCIETY.

SINCE the founding of the Society in 1888, the regular, including the annual, meetings have been held almost without exception in New York City, as the most convenient center for the members living in the eastern states and others who might from time to time attend an eastern meeting. The summer meeting, migratory between limits as far apart as Boston and St. Louis, has afforded an annual opportunity for a fully representative gathering, and provision has been made for the convenience of the central and western members by the founding of the Chicago Section in 1897, the San Francisco Section in 1902, and the Southwestern Section in 1906. The desire has, however, often been expressed that the annual meeting of the Society might, when geographic and other conditions were exceptionally favorable, be occasionally held like that of many other scientific bodies in connection with the meeting of the American association for the advancement of science, a gathering which naturally affords many conveniences of travel and scientific advantages. It was therefore decided to hold the annual meeting of 1908 at Baltimore in affiliation with the Association, the days chosen being Wednesday and Thursday, December 30-31.

Two sessions were held on each day in the Biological Laboratory of Johns Hopkins University. The total attendance numbered about seventy-five, including the following fifty-seven members of the Society :

Miss C. C. Barnum, Dr. E. G. Bill, Professor G. A. Bliss, Professor E. W. Brown, Dr. A. B. Chace, Professor A. S. Chessin, Dr. A. B. Coble, Dr. Abraham Cohen, Professor F. N. Cole, Dr. G. M. Conwell, Miss E. B. Cowley, Mr. F. F. Decker, Mr. C. E. Dimick, Professor John Eiesland, Professor L. P. Eisenhart, Professor J. C. Fields, Dr. Fabian Franklin, Dr. W. A. Granville, Dr. F. L. Griffin, Professor C. C. Grove, Professor Harris Hancock, Professor J. G. Hardy, Professor C. N. Haskins, Rev. A. S. Hawkesworth, Dr. A. A. Himowich, Professor H. L. Hodgkins, Professor L. S. Hulbert, Professor J. I. Hutchinson, Professor Edward Kasner, Profes-

sor C. J. Keyser, Mr. W. J. King, Mr. Joseph Lipke, Dr. W. R. Longley, Professor W. H. Maltbie, Professor Mansfield Merriman, Professor J. F. Messick, Professor G. A. Miller, Professor J. A. Miller, Dr. C. L. E. Moore, Professor Frank Morley, Professor G. D. Olds, Professor N. A. Patillo, Dr. H. B. Phillips, Professor E. D. Roe, Professor F. H. Safford, Miss M. E. Sinclair, Professor P. F. Smith, Professor Virgil Snyder, Professor H. F. Stecker, Professor J. H. Tanner, Professor E. J. Townsend, Professor H. W. Tyler, Professor Anna L. Van Benschoten, Professor C. A. Waldo, Professor H. S. White, Professor T. W. D. Worthen, Professor J. E. Wright.

Wednesday evening was set apart for a social reunion and dinner, at which forty-five members were present. The meeting of the Council was held later on the same evening.

The annual meeting was especially marked as the occasion of the retiring address of President H. S. White on "Bezout's theory of resultants and its influence on geometry," which was delivered at the opening of the afternoon session on Wednesday.

The heavy programme fully occupied the four sessions. President White occupied the chair, being relieved by Professor Morley and Vice-Presidents G. A. Miller and Kasner. The Council announced the election of the following persons to membership in the Society: Professor G. N. Armstrong, Ohio Wesleyan University; Professor P. F. Gaehr, Robert College, Constantinople; Dr. Frank Irwin, Princeton University; Miss Mary E. Wells, Mount Holyoke College. Six applications for membership were received.

The reports of the Treasurer, Auditing Committee, and Librarian will appear in the Annual Register, now in press. The membership of the Society has increased during the past year from 582 to 601, including at present 55 life members. The number of papers presented at all meetings during the year was 169. The total attendance of members at the meetings was 281. The Treasurer's report shows a balance of \$6995.80, of which \$3215.20 is credited to the life membership fund. Sales of the Society's publications during the year amounted to \$1385.36.

At the annual election, which closed on Thursday morning, the following officers and members of the Council were chosen:

<i>President,</i>	Professor MAXIME BÔCHER.
<i>Vice-Presidents,</i>	Professor EDWARD KASNER.
	Professor E. B. VAN VLECK.

<i>Secretary,</i>	Professor F. N. COLE.
<i>Treasurer,</i>	Professor J. H. TANNER.
<i>Librarian,</i>	Professor D. E. SMITH.

Committee of Publication,
 Professor F. N. COLE,
 Professor D. E. SMITH,
 Professor VIRGIL SNYDER.

Members of the Council to serve until December, 1911,

Professor H. B. FINE,	Professor F. R. MOULTON,
Professor O. D. KELLOGG,	Professor E. J. WILCZYNSKI.

The following papers were read at this meeting :

- (1) Professor R. D. CARMICHAEL : "On r -fold symmetry of plane algebraic curves."
- (2) Professor R. D. CARMICHAEL : "A general principle of inversion, with applications."
- (3) Dr. W. R. LONGLEY : "Some sufficient conditions in the theory of implicit functions."
- (4) Dr. C. L. E. MOORE : "Properties of systems of lines in space of four dimensions and their interpretation in circle geometry."
- (5) Dr. F. R. SHARPE : "The topography of the integral curves of a differential equation."
- (6) Mr. JOSEPH LIPKE : "Note on isotropic ruled surfaces."
- (7) Professor JOHN EIESLAND : "On a species of cubic surfaces of the sixth class."
- (8) President E. O. LOVETT : "Integrable problems of three bodies."
- (9) Professor J. I. HUTCHINSON : "On linear transformations which leave an hermitian form invariant."
- (10) President H. S. WHITE :— Presidential Address : "Bezout's theory of resultants and its influence on geometry."
- (11) Professor FRANK MORLEY : "Plane sections of the Weddle surface."
- (12) Professor G. A. MILLER : "Finite groups which may be defined by two operators satisfying two conditions."
- (13) Dr. F. L. GRIFFIN : "Tests comparing the apsidal angles and periodic times for different laws of central force."
- (14) Dr. E. G. BILL : "Existence 'im Kleinen' of a space curve which minimizes a definite integral."

(15) Dr. E. G. BILL: "An a priori existence theorem in three dimensions for the calculus of variations."

(16) Mr. J. R. CONNER: "Curves and surfaces which admit configurations of the Cayley-Veronese type."

(17) Mr. D. D. LEIB: "The complete system of invariants for two triangles."

(18) Dr. W. A. GRANVILLE: "Dual formulas in spherical trigonometry."

(19) Professor C. J. KEYSER: "Concerning euclidean geometries without points and lines."

(20) Miss M. E. SINCLAIR: "The problem of the surface of revolution with two end points variable on circles."

(21) Professor G. A. BLISS: "On the construction of the coordinate system of analytic projective geometry."

(22) Professor EDWARD KASNER: "The group generated by turns and slides."

(23) Professor EDWARD KASNER: "Catenaries in an arbitrary field of force."

(24) Professor C. N. HASKINS: "Numerical computation of reaction velocity constants."

(25) Dr. W. B. CARVER: "Degenerate pencils of quadrics connected with $\Gamma_{n+4, n}^{n+2}$ configurations."

(26) Dr. C. F. CRAIG: "On a class of hyperfuchsian functions."

(27) Professor VIRGIL SNYDER: "Surfaces and congruences derived from the cubic variety having a double line in four-dimensional space."

(28) Professor W. B. FITE: "Irreducible homogeneous linear groups in an arbitrary infinite field."

(29) Dr. ARTHUR RANUM: "On certain solids in riemannian space."

(30) Dr. ARTHUR RANUM: "On the rank of a matrix."

(31) Dr. H. B. PHILLIPS: "Polygons on a quadric surface."

(32) Professor J. G. HARDY: "Note on a theorem of Pierdini concerning four-dimensional curves."

(33) Dr. A. B. COBLE: "Combinants of binary forms."

Mr. Conner and Mr. Leib were introduced by Professor Morley. In the absence of the authors the papers of Dr. Sharpe, Dr. Carver, Dr. Craig, Professor Fite, and Dr. Ranum were read by Professor Snyder, and the papers of Professor Carmichael, President Lovett, and Dr. Phillips were read by title. President White's address will appear in the next num-

ber of the BULLETIN. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. This paper is a generalization of a preceding one on four-fold symmetry of plane algebraic curves. Professor Carmichael shows that there are still two classes of curves distinguished by differing forms of the equation. Taking the general cartesian equation in the form

$$\sum a_{ts}x^t y^s = 0 \quad (t + s \leq n),$$

with origin at the center of symmetry, it is shown that for r -fold symmetry the necessary and sufficient relations among the constants a in the two classes are respectively

$$\sum_{t,s=0}^{t,s=v} a_{ts} \cos^t \theta_1 \sin^s \theta_1 = \sum_{t,s=0}^{t,s=v} a_{ts} \cos^t (\theta_1 + \alpha\phi) \sin^s (\theta_1 + \alpha\phi), \quad (t + s = v),$$

$$\sum_{t,s=0}^{t,s=v} a_{ts} \cos^t \theta_1 \sin^s \theta_1 = (-1)^\alpha \sum_{t,s=0}^{t,s=v} a_{ts} \cos^t (\theta_1 + \alpha\phi) \sin^s (\theta_1 + \alpha\phi), \quad (t + s \neq v),$$

where $v = 0, 1, 2 \dots, n$ (different equations being formed for each value of v), α is any integer $\leq r - 1$, and $\phi = 360^\circ / r$. Methods are given for determining the a 's in each case and the results are worked out for the smaller values of r and n . Finally, an example is given of a special tenfold symmetrical curve.

2. G. Cantor has developed several properties of the function $\beta(N)$ defined by

$$\beta(1) = 1, \quad \beta(N) = (-1)^{\alpha_1 + \alpha_2 + \dots + \alpha_n},$$

where $N = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n} > 1$, $p_1, p_2, p_3, \dots, p_n$ being different primes. Professor Carmichael establishes the following inversion principle :

If n is an integer and $f(n)$ and $F(n)$ are two functions connected by the relation $f(n) = \sum_k F(kn)$, but otherwise quite arbitrary, then $\sum_n F(n^2) = \sum_n \beta(n) \cdot f(n)$, where the summation may extend to an infinite number of terms or may stop short at a finite number of terms by the assumption that $f(n)$ and $F(n)$ have only the value zero when n is greater than some fixed number.

Several applications are made of this principle. In particular it is applied to the summation of certain classes of infinite series.

3. In Dr. Longley's paper the functions considered are real functions of real variables. The following theorem on implicit functions is well known: Consider the system of equations

$$(1) \quad f_i(x_1, \dots, x_n; u_1, \dots, u_m) = 0 \quad (i = 1, \dots, n).$$

Suppose a special solution

$$(2) \quad x_1 = a_1, \dots, x_n = a_n; \quad u_1 = b_1, \dots, u_m = b_m,$$

where a_i and b_i denote constants, is known, and that in the neighborhood of the system of values (2) the functions f_i are of class * C' . Suppose the functional determinant

$$\Delta = \frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_n)} \neq 0$$

for the system of values (2). Then equations (1) have one and only one solution of the form

$$A: \quad x_i = \phi_i(u_1, \dots, u_m),$$

where $a_i = \phi_i(b_1, \dots, b_m)$.

The hypothesis of this theorem states a condition which is sufficient to insure the existence of a unique solution of the type A . The condition $\Delta \neq 0$ is however not necessary. Supposing that $\Delta = 0$, it is the object of this paper to present some other conditions which are sufficient to insure the existence of a unique solution of the type A . The geometric significance of some of the theorems is considered. Applications are made to the existence, in the plane and in space, of a field of extremals about a point where all the extremals of the set pass through that point.

4. If the equations of a line in a space of four dimensions S_4 be written in the form

$$x_1 = a_1 + \alpha_1 z, \quad x_2 = a_2 + \alpha_2 z, \quad x_3 = a_3 + \alpha_3 z, \quad x_4 = z,$$

* That is continuous with continuous first partial derivatives.

then the equations of the tangent plane to a ruled surface in S_4 may be written

$$\frac{x_1 - a_1 - \alpha_1 x_4}{x_2 - a_2 - \alpha_2 x_4} = \frac{a_1' + \alpha_1' z}{a_2' + \alpha_2' z} = \rho_1,$$

$$\frac{x_1 - a_1 - \alpha_1 x_4}{x_3 - a_3 - \alpha_3 x_4} = \frac{a_1' + \alpha_1' z}{a_3' + \alpha_3' z} = \rho_2.$$

Considering $a_1', a_2', a_3', \alpha_1', \alpha_2', \alpha_3'$ as the homogeneous coordinates of the projectivity between the points z of any generator and the planes passing through the generator and tangent to the ruled surface, Dr. Moore discussed the general properties of lines in space of four dimensions and interpreted the results in circle geometry in space of three dimensions.

5. Dr. Sharpe's paper discusses the topography of the integral curves of $dy/dx = c_1/c_2$ where $c_1 = 0, c_2 = 0$ are conics. The nature of the integral curves depends on the locus of points of the conic $c_1 = c_2 \lambda$ at which the conic has the slope λ . This locus is also the locus of the inflections of the integral curves and is a quintic having a double point at each of the four intersections of $c_1 = 0, c_2 = 0$ and passing through the three intersections of the three pairs of lines through the four points. When the conics $c_1 = 0, c_2 = 0$ have simple contact, the quintic has a triple point at the point of contact. When the contact is of the second order, the triple point consists of a cusp and a simple branch. When the contact is of the third order, the quintic degenerates into the common tangent and a quartic having a tacnode at the point of contact. Examples are given of the most interesting cases.

6. In discussing the problem of the line of striction on a ruled skew surface, it is generally shown that an indeterminacy arises if the generator is a minimal straight line. In this note, Mr. Lipke defines the line of striction in one of the usual ways and, introducing infinitesimals of the second order, gets a definite curve on the isotropic skew surface as its line of striction. He further discusses the relations existing between this curve, the edge of regression of the minimal developable enveloped by the system of minimal planes through the generators, and the focal curve of the congruence of normals to the surface.

7. Professor Eiesland's paper is a study of cubic surfaces of the form

$$A + Bx + Dy + Cz + Exz + Fxy + Gyz + Hxyz = 0,$$

with the identical relation $EGAF = HDCB$ between the coefficients. It is shown that two pairs of families of twisted cubics exist on it such that by the involutory transformation

$$\bar{x} = \frac{\lambda}{x}, \quad \bar{y} = \frac{\mu}{y}, \quad \bar{z} = \frac{\nu}{z},$$

where λ, μ, ν are properly chosen values, each pair is transformed into the other. These surfaces are connected with a unicursal quartic in the plane at infinity.

8. Generalizing a well-known problem of Bertrand in mechanics, Oppenheim showed in 1894* that if three given particles describe under central conservative forces three given plane curves, the forces are derived from a potential function which may be written in a form in which there appear only the constant of areas, the masses of the bodies, their coordinates referred to the center of gravity of the system, and the first partial derivatives of the functions defining the orbits. Oppenheim applied his results to the rediscovery of the known solutions of the plane problem of three bodies in which the law of force is as the masses, and either directly as the distance or inversely as the square of the distance.

In Professor Lovett's paper, by the aid of the theory of partial differential equations of the first order and the form of the force function revealed in Oppenheim's memoir, there is constructed an infinitude of integrable problems of three bodies in the plane under forces depending on the masses and mutual distances of the bodies. These problems are grouped in four families according to the form of the potential function. The paper concludes with a fifth family in which the coordinates of the bodies appear explicitly in the force function.

9. The object of the paper by Professor Hutchinson is to give some of the properties of linear transformations on n variables for which an hermitian form H is invariant. The substitutions whose characteristic determinants have no equal roots are classified into elliptic (all roots unimodular) or hyperbolic of q differential types (when $2q$ of the roots are not unimodular).

* *Publicationen der v. Kuffner'schen Sternwarte*, vol. 3, pp. 71-95.

The fixed points of an elliptic substitution are situated so that p of them are inside and $n - p$ outside the locus $H = 0$ (p being the number of positive terms of H and $p \leq n - p$). The fixed points for a hyperbolic substitution of the q th type include q pairs which are on H , $p - q$ points inside H , and $n - p - q$ outside H ($q \leq p$). The generalized Poincaré series is considered, and the form of its expansion in the vicinity of a fixed point determined.

11. The plane section of a Weddle surface is not an arbitrary quartic curve, but one for which an invariant vanishes. The curve contains a configuration B^6 , namely, where it is cut by the lines on the surface. The theorem proved by Professor Morley is that it contains an infinity of such configurations.

12. It is evident that two or more distinct operators, none of which is identity, cannot be sufficiently restricted by a single condition of the form $s_1^\alpha s_2^\beta s_1^\gamma s_2^\delta \cdots = 1$, where $\alpha, \beta, \gamma, \delta, \dots$ are positive or negative integers, to generate only groups whose orders are limited by the values of $\alpha, \beta, \gamma, \delta, \dots$. On the other hand it is possible to find pairs of such conditions which are such that each pair is satisfied by two generators of only a limited number of groups. Such a pair cannot be of the form $s_1^\alpha = 1, s_1^\beta = 1$ since the order of the product of two operators is not limited by the orders of these operators. In the present paper Professor Miller considers only those cases where a pair of conditions between two operators s_1, s_2 determines either only a single group or a limited number of groups.

If the two conditions are of the form $s_1^n = 1, s_1^\alpha = s_2^\beta$ the order of G is limited when n and α are relatively prime, and only then. When these conditions are satisfied, G is cyclic and its order divides βn . For instance, the cyclic group of order 40 is completely defined by the two conditions $s_1^{10} = 1, s_1^3 = s_2^4$. On the other hand, the two conditions $s_1^{10} = 1, s_1^4 = s_2^3$ are satisfied by two generators of an infinite number of distinct groups. The following theorem is useful in the study of some of the pairs of conditions: If two commutative operators satisfy the condition $s_1^n = s_2^m$, they generate the direct product of two cyclic groups whose orders are respectively the lowest common multiple of the orders of these operators, and a divisor of the highest common factor of α, β . The latter group may be identity, and it must be identity whenever α, β are relatively prime.

A large number of interesting results are obtained when the two conditions are of the form $s_1^\alpha = 1$, $s_1 s_2 = s_2^\beta s_1^\beta$. Among these are the following: These two conditions are satisfied by the two generators of only a finite number of groups when $\beta = 1$ and $\alpha \neq 1$, but they are satisfied by two generators of each one of an infinite system of distinct groups when also $\alpha = 1$. If two generators of a group satisfy the conditions $s_1^3 = 1$, $s_1 s_2 = s_2 s_1^{-1}$, the group must be either tetrahedral or the group of order 24 involving no subgroup of order 12.

13. In an earlier paper* Dr. Griffin proved certain theorems concerning the apsidal angle Θ in central orbits, obtaining, for example, conditions upon any given law of force necessary and sufficient in order that $\Theta > \pi$ in all orbits of any region. An examination of the method of proof, which consisted of a comparison of the definite integrals giving Θ and π (the corresponding angle for Newton's law), suggests the possibility of generalizing the theorems so as to compare the values of Θ in orbits under any two arbitrarily chosen laws of force.

Several such generalizations are given in the present paper, and applications are made to various simple laws of force. The new proofs, while more involved than those of the former paper, proceed along the same general lines. The criteria again are very simple and easily applied.

Analogous theorems are established for the periodic times in the orbits.

14. Hilbert in 1899 stated that under certain conditions a solution of the calculus of variations problem always existed, and later sketched a proof of this statement which Professor Bolza has since completed.† In this proof the integrand of

$$\int_{t_0}^{t_1} F(x, y, x', y') dt$$

is defined only for a finite closed region and the minimizing curve is built up as the limiting position of an "infinite" sequence of division points of a set of curves which approximate to giving the integral its minimum value.

In Dr. Bill's papers the similar question is considered for the integral

* Cf. BULLETIN, vol. 14 (Oct., 1907), pp. 6-16.

† Bolza, Lectures on the Calculus of Variations, p. 245.

$$J = \int_{t_0}^{t_1} F(x, y, z, x', y', z') dt,$$

carried over a curve in space.

In his first paper it is shown that a solution "im Kleinen" exists, the proof depending on the existence of a field of extremals through a given point.

15. In his second paper Dr. Bill proves the existence "im Grossen," by a method whereby the minimizing curve is obtained by applying the existence theorem "im Kleinen" to a "finite" number of points which are defined as limiting points of points lying on a sequence of approximating curves. Considerable simplification over the original form of the method of Hilbert is thus obtained.

16. Mr. Conner's paper is in abstract as follows: In a space of four dimensions, lines bisecant to a curve C_p^m , of order m and genus p , lie on a 3-way spread Λ of order $h = \frac{1}{2}(m-2)(m-3) - p$, in which C_p^m is an $(m-2)$ -fold curve. A space α meets Λ in a surface K_h^m of order h . Denote by Δ_m and Γ_σ , respectively, the plane section of a complete m -point in three dimensions, and the space section of a complete σ -point in four dimensions. σ points on C_p^m define a configuration Γ_σ inscribed in K_h^m . A plane section of Λ admits Δ_m 's determined by the sets of m points in which spaces on the plane meet C_p^m . C_0^4 gives rise to a 4-nodal cubic surface K_3^4 . The theory of the K_3^4 is connected with that of the rational space quartic curve, the underlying quartic being here the (class) quartic in which the tangential developable to C_0^4 meets α . The surface K_5^5 is the general quintic surface with five triple points. The surface K_6^5 has besides five triple points a double curve of order 7. Curves C_p^m in five dimensions determine in a similar way space curves admitting Γ_m 's.

17. In this paper Mr. Leib derives a complete system of the invariants of two triangles, and proves its completeness. Various relations are discussed, including the condition on two triangles so that a conic may be inscribed in one and circumscribed about the other. This leads at once to the equation of the locus of the foci of parabolas through three points, and properties of this locus are deduced directly from the invariant form. The covariants of special interest are also discussed.

18. As generally given by writers on spherical trigonometry, the formulas for the sides of a spherical triangle in terms of the angles have had a certain resemblance to the formulas giving the angles in terms of the sides, but the correspondence has not been perfect. Möbius was the first to point out the fact that if the supplements of the angles of the triangle were used, instead of the angles themselves, one set of these formulas is the dual of another. He did not, however, publish any such sets of formulas, and in this paper Dr. Granville shows what the ordinary formulas of spherical trigonometry become when given in this form. The following theorem is first established :

In any relation between the parts of a general spherical triangle, each part may be replaced by the supplement of the opposite part, and the relation thus obtained will hold true.

When the above theorem is applied to a relation involving one or more of the sides and the supplements of the angles of a spherical triangle, we get the following principle of duality :

If the sides of a general spherical triangle are denoted by the Roman letters a, b, c , and the supplements of the corresponding opposite angles by the Greek letters α, β, γ , then from any given formula involving any of these six parts, we may write a dual formula by simply interchanging the corresponding Greek and Roman letters. The old and the new sets of formulas are then compared, and spherical triangles are solved using both.

19. A system of postulates equivalent to the system (H) of Hilbert's Foundations or to Veblen's system (V) of geometric axioms (*Transactions*, volume 5)—happily described by the latter as categorical—define, not a class s of objects satisfying the system, but a class or category C of such classes s . C is determined by various systems, and is composed of manifold classes s . To each s corresponds a geometric theory determined by any one of the systems and the same for all of them. These theories, though one logically, are many psychologically. Hence the interest attaching to study directed upon the varied content as distinguished from the definition of C . If e, e', e'' denote the undefined elements of (H), e will represent the undefined element of (V), e' and e'' being in the latter system defined by means of the undefined notion of order. The problem is to find near-lying interpretations of the elements, *i. e.*, to discover such components s of C as are easily accessible and

hence interesting to intuition. After the usual interpretation, the most familiar one is that in which s is ordinary space bereft of a point O , e means point, and e' and e'' denote respectively the circles and spheres that would contain O if it were not excluded. In Professor Keyser's paper the following classes s were indicated: (1) s is ordinary space (of planes), lacking any chosen sheaf or bundle o of planes, e denotes ordinary plane, e' denotes pencil of planes lacking the plane belonging to o , and e'' denotes sheaf of planes lacking the e' belonging to o ; (2) s is the same as before, e denotes sphere containing the vertex P of o , e' denotes sphere range (or circle) determined by two e 's, and e'' denotes sphere congruence (or circle congruence) determined by three e 's not belonging to a same e' . The (H) and (V) axioms for two dimensions are satisfied by the following: (1) s is the ordinary plane of lines lacking any chosen pencil o , e denotes line, and e' denotes pencil lacking the e of o ; (2) s is same as before, e is circle through the vertex P of o , and e' is circle range determined by two e 's; (3) s is a sheaf of planes minus a pencil o , e is plane, and e' is pencil of planes minus the e belonging to o ; (4) s is a sheaf of lines lacking a pencil o , e is line, and e' is pencil of lines minus the e of o . Another s is a sheaf of circular cones of common vertex minus a pencil of them.

20. Miss Sinclair's paper is in abstract as follows: The original Plateau experiment of a catenoid film between two anchor rings is the problem of the surface of revolution of minimum area in case the two end points are variable on two circles equal and equidistant from the axis of revolution. Let the x -axis be the axis of revolution, and the y -axis pass through the center of the first circle. Let ρ be the radius of the circles, d the distance of their centers from the axis, (κ, a) the vertex of the catenary. Then we have the equations

$$(1) \quad x_0 = \rho \cos \theta, \quad y_0 - d = \sin \theta;$$

$$(2) \quad y_0 = aC(u_0), \quad x_0 = au_0 + \kappa;$$

$$(3) \quad 1 - \frac{x_0}{y_0 - d} S(u_0) = 0,$$

which are sufficient to determine x_0, y_0, a, κ if u_0 is known. On account of the symmetry of the problem, x_1, y_1 can be found from (2) by replacing u_0 by $-u_0$.

The parameter u_0'' of the point focal to u_0 is the positive solution of

$$(4) \quad \frac{\rho}{\alpha} = C^2(u_0)S(u_0) \frac{\phi(u_0'') - \phi(u_0)}{\phi(u_0'') + u_0 + C(u_0)S(u_0)},$$

where

$$\phi(u_0) = \frac{C(u_0)}{S(u_0)} - u_0.$$

The value of u_0 is restricted by the conditions for fixed end points, on account of the symmetry of the problem, to the interval $(u_0^*, 0)$ where u_0^* is the negative solution of $\phi(u_0) = 0$. The Bliss condition for two variable end points requires for one problem that $\phi(u_0'') \leq 0$. For the particular value u_0'' for which $\phi(u_0'') = 0$ we obtain, from the above condition,

$$(5) \quad \frac{\rho}{d} = \frac{-C(u_0)S(u_0)\phi(u_0)}{[1-S(u_0)]u_0 + 2C(u_0)S(u_0)}.$$

Moreover, ρ/d is a continuous function of u_0 in the interval $(u_0^*, 0)$, increasing from 0 to ∞ . Hence if ρ and d are given, (5) may on this interval be solved uniquely for u_0 . Let this solution be \bar{u}_0 . If ρ and d are constant, u_0'' is found to be an increasing function of u_0 , and the Bliss condition is satisfied for $u_0 > \bar{u}_0$.

A physical illustration is of interest and the results of experiment are here given :

$$\begin{aligned} \rho &= 2.5 \text{ cm.}, \quad \theta = 33^\circ, \\ d &= 5 \text{ cm.}, \quad \text{length of film} = 2\kappa = 3.08. \end{aligned}$$

The corresponding theoretical results are (ρ and d being given)

$$\begin{aligned} \rho &= 2.5 \text{ cm.}, \quad \theta = 33^\circ 0' 30'', \\ d &= 5 \text{ cm.}, \quad 2\kappa = 3.01. \end{aligned}$$

21. In Professor Bliss's paper it is shown how the projective coordinates of points and lines in the plane may be defined in terms of anharmonic ratios, and how the relation

$$u_1x_1 + u_2x_2 + u_3x_3 = 0$$

expressing the incidence of the point (x_1, x_2, x_3) and the line (u_1, u_2, u_3) may be proved directly from the properties of an-

harmonic ratios. The definition of the coordinates is not new, but the proof of the incidence relation seems more direct than those usually given.

22. The transformations considered in Professor Kasner's first paper operate on the oriented lineal elements of the plane. By a turn T_α each element rotates about its own point through a fixed angle α . By a slide S_κ each element moves along its own line through a fixed distance κ . It is shown that every combination of turns and slides is reducible to the form $T_\alpha S_\kappa T_\beta$, so that a continuous group of three parameters is obtained. The only contact transformations included are the dilatations. Applied to a simply infinite system of curves the turns lead to the familiar isogonals and the slides to the equitangentials whose theory has been developed recently by Scheffers. The combination of both constructions leads to ∞^3 related systems (though in special cases, admitting automorphic transformations, there will be fewer). The two known theories of trajectories, which appear isolated in the usual treatment, are shown to be special cases of one general theory. The fundamental figure considered, which includes the point, line, and circle as special cases, is formed of ∞^1 elements whose points are on a circle and whose directions are equally inclined to that circle. It is termed a geometric *turbine*. The most general transformation of turbines involves 15 parameters. The resulting geometry is isomorphic with projective geometry in three dimensions. Many previous theories appear in a new light.

23. The study of the equilibrium of a homogeneous flexible inextensible string in a given field of force leads to ∞^5 curves in space, or ∞^3 in the plane. These general systems of catenaries are explored in Professor Kasner's second paper. The results are analogous to but distinct from those dealing with dynamical trajectories (*Transactions*, 1906, 1907). Characteristic properties are obtained, largely by intrinsic methods. It is shown that the field of force may be constructed when a sufficient (infinite) number of its catenaries are known. The modifications necessary for non-homogeneous strings are considered.

24. The paper of Professor Haskins has reference to a problem of chemical dynamics. The course of an incomplete bimolecular reaction is determined by the differential equation

$$dx/dt = k_1(a_1x^2 + 2b_1x + c_1) - k_2(a_2x^2 + 2b_2x + c_2),$$

where $a_1, b_1, c_1, a_2, b_2, c_2$ are constants known from the initial conditions, and k_1, k_2 are the so-called "reaction constants" which are to be determined from simultaneous observations of x and t . In the reactions hitherto studied the ratio k_1/k_2 has been found from the equilibrium conditions ($t = \infty$) and in this case the computation of k_1/k_2 offers no difficulty. In recent investigations of gas reactions the equilibrium conditions were not attainable, and a method of determination of k_1, k_2 from any two pairs of values of x and t became necessary. The determination can be reduced to the solution of the equation in γ

$$\log \frac{\gamma + z_2}{\gamma - z_2} - \frac{t_2}{t_1} \log \frac{\gamma + z_1}{\gamma - z_1} + \left(\frac{t_2}{t_1} - 1 \right) \log \frac{\gamma + z_0}{\gamma - z_0} = 0,$$

$$(z_0 < z_1 < z_2, \quad t_1 < t_2),$$

and in important special cases to that of

$$\log \frac{\gamma + z_2}{\gamma - z_2} - \frac{t_2}{t_1} \log \frac{\gamma + z_1}{\gamma - z_1} = 0, \quad (z_1 < z_2, \quad t_1 < t_2).$$

A single condition must be satisfied in order that a root may exist, and the root if existent is unique. The numerical solution proceeds by iteration and converges in general with considerable rapidity.

25. In a paper presented to the Society at the December meeting, 1906 (which has since appeared in the January number of the *American Journal of Mathematics*), Dr. Carver called attention to certain pencils of quadrics connected with the configuration $\Gamma_{n+4, n}^{n+2}$. The purpose of the present note is to investigate the question of the existence of degenerate pencils for configurations which are not degenerate.

26. Alezais (*Annales de l'Ecole normale supérieure*, (3), volume 19 (1902), page 261) has studied a monodromy group of a particular Riemann surface. Dr. Craig's paper studies a simpler group and functions belonging to the group. An extension of a known result on the number of linearly independent moduli belonging to an abelian integral of the first kind on a

binomial Riemann surface, and the upper limit for the number of generators of the maximum monodromy group connected with such surfaces are obtained.

27. By the same methods as were employed in the study of cubic varieties having nine double points (abstract in the BULLETIN, volume 15, page 67) Professor Snyder examines all the forms having one or more double lines, filling in the details outlined by the Segre memoirs and supplying a large number of particular cases. All the particularizations of the $(3, 2)$ congruence having a focal line can be obtained and classified from this standpoint, and also the forms of the $(2, 2)$ congruences, the latter being already known. Finally, a large number of surfaces are obtained which remain invariant under birational transformations of infinite order.

28. The principal result in Professor Fite's paper is the following theorem: A necessary and sufficient condition that any group of finite order be simply isomorphic with an irreducible group in any infinite field is that its central be cyclic (or identity). As a direct application of this theorem, the number of those substitutions of finite order that are commutative with a given substitution that is irreducible in any infinite field is determined.

29. Defining a parallelogram in riemannian space as a portion of a Clifford surface bounded by a skew quadrilateral whose opposite sides are right parallels and left parallels, respectively, and a parallelepiped as a solid bounded by three pairs of opposite faces, all of which are parallelograms, Dr. Ranum finds that there are two distinct species of parallelepipeds, one depending on six parameters and the other on two parameters. Defining a quasi-parallelogram as a plane quadrilateral whose opposite sides are congruent, and a quasi-parallelepiped in a corresponding manner, he shows that while there exist oblique quasi-parallelograms (having no lines of symmetry), nevertheless oblique quasi-parallelepipeds (having no planes of symmetry) do not exist.

30. In order to determine the rank of a matrix it is sometimes of practical utility to know the minimum number of determinants of a given order which must be examined as to their vanishing or non-vanishing. In this paper Dr. Ranum con-

siders a matrix having m rows and n columns, and under certain hypotheses as to the non-vanishing of one or more r -rowed determinants of the matrix finds the minimum number of s -rowed determinants ($s > r$) whose vanishing will insure the vanishing of all the other s -rowed determinants.

31. The polygons of Dr. Phillips's paper are formed of groups of $2n$ points joined in order by generators of a quadric surface. From a remark of Salmon it follows that if each odd point is joined to the non-adjacent even points the $n(n-2)$ lines lie on a surface of class $n-2$. The object of the paper is a discussion of these surfaces.

32. Professor Hardy's paper takes up a theorem of Pironcini concerning "cylindrical helices" in a space of four dimensions and shows that under Pironcini's assumptions such curves lie entirely in a properly chosen three dimensional space, and are not, properly speaking, four-dimensional curves.

33. The involution $I_{\kappa, n-\kappa}^n$ of ∞^κ sets of n values such that κ values in a set determine the remaining $n-\kappa$ is expressed analytically by a *special* symmetric binary form in $\kappa+1$ variables to the order $n-\kappa$,

$$I_{\kappa, n-\kappa} \equiv (a_1 x_1)^{n-\kappa} (a_2 x_2)^{n-\kappa} \dots (a_\kappa x_\kappa)^{n-\kappa} (a_{\kappa+1} x_{\kappa+1})^{n-\kappa} = 0.$$

For assigned values of $x_1, x_2, \dots, x_\kappa$, $n-\kappa$ values of $x_{\kappa+1}$ are determined, the equation $I_{\kappa, n-\kappa} = 0$ being satisfied by any $\kappa+1$ values of the whole set of n . Such a set of n values with reference to a *general* symmetric form may be called an "involution set." The object of Dr. Coble's paper is to determine the number of such involutive sets possessed by the general symmetric form and to determine the conditions upon the form in order that this number shall increase until it attains its maximum, ∞^κ .

A geometrical representation of the form as a spread of order $\kappa+1$ and dimension $n-\kappa-1$ in an $S_{n-\kappa}$ is given. The involutive sets then appear as "apolar n -points" of the spread which lie on a norm-curve $N_{n-\kappa}$. The definition of an apolar n -point is a set of n points any $\kappa+1$ of which are apolar to the spread of order $\kappa+1$.

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