The report contains chapters on the following topics: the asymptotic properties of ternary homogenous forms; the penultimate forms of a curve; the existence of "free summits" on a degenerate algebraic curve; the classification of curves on the basis of the penultimate forms. Further investigations are in progress. The memoir, as a whole, will be offered for publication the coming year.

F. N. COLE,
Secretary.

THE APRIL MEETING OF THE CHICAGO SECTION.

THE twenty-third regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of Chicago, on Friday and Saturday, April 17–18, 1908. One session was held on Friday and two on Saturday. On Friday evening twenty of the members dined together informally in the private dining room of the University of Chicago Commons. The attendance at the three sessions included forty-five persons, among whom were the following thirty-three members:

Dr. G. D. Birkhoff, Professor Oscar Bolza, Dr. R. L. Börger, Professor H. E. Buchanan, Mr. Thomas Buck, Professor W. H. Bussey, Dr. A. R. Crathorne, Professor D. R. Curtiss, Professor L. E. Dickson, Mr. Arnold Dresden, Mr. E. P. R. Duval, Professor W. B. Ford, Mr. R. M. Ginnings, Professor Harriet E. Glazier, Professor C. N. Haskins, Mr. T. H. Hildebrandt, Mr. F. H. Hodge, Dr. A. C. Lunn, Mr. W. D. MacMillan, Professor G. A. Miller, Dr. J. C. Morehead, Professor F. R. Moulton, Dr. L. I. Neikirk, Professor H. L. Rietz, Miss Ida M. Schottenfels, Mr. A. R. Schweitzer, Professor H. E. Slaught, Dr. Clara E. Smith, Dr. A. L. Underhill, Professor E. B. Van Vleck, Dr. A. E. Young, Professor J. W. A. Young, Professor Alexander Ziwet.

Professor G. A. Miller, Vice-President of the Society and chairman of the Section, presided at all of the sessions. In opening the meeting he spoke of the great loss sustained by the Society in the recent death of Professor Heinrich Maschke. A committee, consisting of Professors E. B. Van Vleck, Alexander Ziwet, and H. E. Slaught, was appointed to draft suitable resolutions on behalf of the Section, and these were presented at the close of the session on Saturday morning, adopted by a rising vote, and ordered spread upon the minutes. The resolutions are printed immediately following this report. A memorial of Professor Maschke, prepared by Professor Bolza, will soon appear in the BULLETIN.

The following papers were read :

(1) Dr. C. H. SISAM : "On a locus determined by concurrent tangents."

(2) Professor W. B. FORD : "On the integration of the equation $a_0(x)\ u(x+2) + a_1(x)\ u(x+1) + a_2(x)\ u(x) = 0$."

(3) Professor D. R. CURTISS : "On the real branches of implicit functions in the neighborhood of multiple points."

(4) Mr. L. L. DINES : "A method of investigating numbers of the forms $6^\kappa \cdot s \pm 1$."

(5) Professor L. E. DICKSON : "Criteria for the irreducibility of a reciprocal equation."

(6) Professor L. E. DICKSON : "On reciprocal abelian equations."

(7) Professor L. E. DICKSON : "On the congruence $x^n + y^n + z^n \equiv 0 \pmod p$."

(8) Professor JACOB WESTLUND : "Note on the equation $x^n + y^n = nz^n$."

(9) Mr. F. H. HODGE and Mr. E. J. MOULTON : "On certain characteristics of orbits for a general central force."

(10) Professor G. A. MILLER : "The central of a group."

(11) Dr. A. E. YOUNG : "On the problem of the spherical representation and the characteristic equations of certain classes of surfaces."

(12) Dr. A. C. LUNN : "A continuous group related to von Seidel's optical theory."

(13) Dr. A. C. LUNN : "A minimal property of simple harmonic motion."

(14) Dr. A. C. LUNN : "The deduction of the electrostatic equations by the calculus of variations."

(15) Mr. A. R. SCHWEITZER : "Remark on Enriques's review of the foundations of geometry."

(16) Mr. A. R. SCHWEITZER : "On the calculi of relations, classes, and operations."

(17) Professor E. J. WILCZYNSKI : "Projective differential geometry of curved surfaces, fourth memoir."

(18) Dr. G. D. BIRKHOFF : "Irregular integrals of ordinary linear differential equations."

(19) Professor R. D. CARMICHAEL: "On the general tangent to plane curves."

(20) Professor R. D. CARMICHAEL: "On plane algebraic curves symmetrical with respect to each of two rectangular axes."

(21) Professor O. D. KELLOGG: "Note on the geometry of continuously turning curves."

(22) Dr. I. SCHUR: "Beiträge zur Theorie der Gruppen linearer homogener Substitutionen."

(23) Mr. W. D. MACMILLAN: "On the character of the solutions of homogeneous linear equations with periodic coefficients."

(24) Mr. A. R. SCHWEITZER: "On the quaternion as an operator in Grassmann's extensive algebra."

Mr. Dines was introduced by Professor Curtiss. Dr. Schur's paper was communicated to the Society through Professor Dickson. In the absence of the authors the papers of Dr. Sisam, Professor Westlund, Professor Wilczynski, Professor Carmichael, and Dr. Schur were read by title. Professor Dickson's first paper appears in the present number of the BULLETIN. His second paper has been published in the April number of the *American Mathematical Monthly*. Abstracts of the other papers follow below, the numbering being the same as that attached to the titles in the above list.

1. In the plane of an arbitrary algebraic curve C there exists a second curve C' such that the points of tangency of three or the tangents to C from each point of C' are collinear. Dr. Sisam's paper is devoted to the discussion of the locus C' and of the dual locus.

2. Professor Ford's paper considers the behavior for large values of x of the general solution of the equation

$$a_0(x)\,u(x+2) + a_1(x)\,u(x+1) + a_2(x)\,u(x) = 0,$$

the coefficients $a_0(x)$, $a_1(x)$, $a_2(x)$ being given functions (real or complex) of x defined for all positive integral values of x sufficiently large. Especial attention is given to the important case in which the coefficients are developable in Maclaurin series about the point $x = \infty$. The results obtained may be employed in the study of the convergence of algebraic continued fractions, as will appear subsequently. The paper has been offered for publication in the *Transactions*.

3. In the paper of Professor Curtiss conditions for the existence of real branches of an implicit function $f(x, y) = 0$, in the neighborhood of a multiple point, are obtained by methods which are analogous to those used by Stolz and others in the theory of maxima and minima, and which constitute a generalization of the method of Dini for an ordinary point. The invariant character of these conditions is discussed, and the connection of semi-definite cases with the contact of branches of the curves

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$$

is indicated.

4. Mr. Dines's paper describes a method of determining factors of numbers of the forms $6^\kappa \cdot s \pm 1$ or primes of those forms, easily applicable to numbers above the limits of the existing factor tables. The x_κ and x'_κ functions of any prime p are defined respectively as the least positive solutions of the congruences $6^\kappa \cdot s - 1 \equiv 0 \pmod{p}$ and $6^\kappa \cdot s + 1 \equiv 0 \pmod{p}$, and recursion formulas are developed for computing the functions. By means of a table of these functions for successive primes, it is possible to deplete by a sieve process analogous to the "sieve of Eratosthenes," any series of consecutive integers, so that any integer remaining, if substituted for s will make $6^\kappa \cdot s + 1$ (or $6^\kappa \cdot s - 1$) prime. The paper will be offered to the *Annals of Mathematics*.

7. In the congruence $x^n + y^n + z^n \equiv 0 \pmod{p}$, considered by Professor Dickson, n and p are assumed to be odd prime numbers. As there are obviously solutions prime to p when $p - 1$ is not divisible by n and when $p - 1$ is a multiple of $3n$, it is assumed that $p = mn + 1$, where m is not divisible by 3. In the second supplement to his Théorie des Nombres, Paris, 1825, Legendre has treated the cases $m = 2, 4, 8, 10, 14, 16$, each by a separate discussion; but overlooked the exceptional character of $n = 3$ when $m = 10$ or 14. Professor Dickson develops a method which applies, without a separation of cases, to the values of m considered by Legendre, and obtains the following result. The congruence has no set of solutions prime to p when n and $p = mn + 1$ are odd primes and $m = 2, 4, 8, 10, 14, 16, 20, 22, 26, 28, 32, 40, 56, 64$, apart from the following exceptions: $m = 10, 14, 20$, $n = 3$; $m = 22$, $n = 3, 31$; $m = 26$, $n = 3, 5$; $m = 28$, $n = 7$; $m = 32$, $n = 3$; $m = 40$,

$n = 7$; $m = 56$, $n = 5$, 11, 17, 113, 227; $m = 64$, $n = 3$, 7, 229, 337, 757. The results are applied to Fermat's last theorem to show that $x^n + y^n + z^n = 0$ has no integral solutions, each not divisible by n, for n an odd prime < 1700, the previous limits being 100 (Sophie Germain), 200 (Legendre), 223 (Maillet), 257 (Mirimanoff). The method is due to Sophie Germain and requires the determination of a prime $p = mn + 1$ for which the above congruence has no solutions prime to p and such that p is not a divisor of $m^m - 1$. The latter number was completely factored by Legendre for the values m which he treated. Part of this work may be avoided. For example, if m is a multiple of 4, the quadratic reciprocity theorem shows that a prime divisor $mn + 1$ (n odd) of $m^m - 1$ must divide $m^{\frac{1}{2}m} - 1$. Additional reduction of the exponent may be made when m is a power of 2.

In the second part of the paper, n is given a fixed value and it is proposed to determine the primes $p = mn + 1$ for which the congruence has no integral solutions prime to p. A general method of attack is developed and the analysis completed for the cases $n = 3, 5, 7$, the number of values of p being 2, 4, 4 respectively.

The first part has been offered for publication in the *Messenger of Mathematics* and the second part in *Crelle's Journal.*

8. In Professor Westlund's note it is proved that the equation $x^n + y^n = nz^n$ is not solvable in rational integers if n is any odd prime and z is not divisible by n.

9. The orbits of particles moving under central forces varying as the first power and as the inverse second, third, and fifth powers have been discussed in detail by many writers, and other special powers also have received attention. The paper of Mr. Hodge and Mr. Moulton considers certain general characteristics of orbits for forces varying as any integral power of the distance, by discussing a transformation of the energy integral expressed in polar coordinates. Without knowing the equations of the orbits, the existence of apses is shown, the aphelian and perihelian distances are found, and certain other properties are deduced.

It is found that for attractive forces the law of the inverse second power gives an intermediate case between two general classes of orbits; the one where the orbits pass through either the center of force or infinity; the other where the orbits have finite perihelion and aphelion distances greater than zero. The paper will be published in the *American Mathematical Monthly.*

10. The totality of the invariant operators of any group G constitutes a characteristic subgroup K known as the central, or the cogredient subgroup of G. When G is abelian, it coincides with K, and when G contains only one invariant operator, K is identity. In these extreme cases the concept of a central does not simplify the considerations with respect to G, but in all other cases it is useful. The quotient group of G with respect to K is the group of cogredient isomorphisms of G and is represented by I_1. The main object of Professor Miller's paper is to determine some of the properties of G from known properties of I_1. The following theorems are proved: If I_1 involves an invariant operator of order n, G contains an invariant commutator of order n and it also has a cyclic quotient group of this order. Moreover, the order of G is divisible by n^3, and I_1 involves two independent operators of order n. The number of Sylow subgroups of any order in G is exactly the same as the number of the Sylow subgroups of the corresponding order in I_1. When all the Sylow subgroups of I_1 are cyclic, every abelian subgroup of I_1 corresponds to an abelian subgroup of G and vice versa. If the order of such an I_1 is $p_1^{\alpha_2}p_2^{\alpha_2}\cdots p_m^{\alpha_m}$, where p_1, $p_2, \cdots, p_m$ are distinct primes in descending order of magnitude, and if I_1 contains a cyclic subgroup of order $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_\gamma^{\alpha_\gamma}$, then G contains a characteristic cyclic subgroup of the same order. The paper will be offered for publication in the *Transactions.*

11. For any surface S referred to lines of curvature the equations of Gauss and Codazzi may be written in the form

$$\frac{\partial M}{\partial v} - \frac{\partial N}{\partial u} = PQ, \quad \frac{\partial P}{\partial v} + MQ = 0, \quad \frac{\partial Q}{\partial u} - NP = 0,$$

where the expression for the corresponding linear element on the gaussian sphere is

$$\overline{d\sigma}^2 = P^2du^2 + Q^2dv^2.$$

Any other equation involving one or more of the functions, taken with the three equations above, determines a certain system of lines on the sphere. We may say therefore that this new equation is characteristic of these lines, or rather of the system of surfaces which have them as the spherical representation of their lines of curvature.

From this standpoint Dr. Young has considered the systems of surfaces defined by the following equations:

$$(1)\ M=0,\quad (2)\ M+N=0,\quad (3)\ \frac{\partial M}{\partial u}+\frac{\partial N}{\partial v}=0,$$

$$(4)\ \frac{\partial^2 \log M}{\partial u\,\partial v}+MN=0,\quad (5)\ \frac{\partial^2 \log MK}{\partial u\,\partial v}-K=0,$$

where

$$K=-\frac{\partial^2 \log M}{\partial u\,\partial v}-MN, \text{ etc.}$$

12. In this paper Dr. Lunn shows how von Seidel's paraxial equations for a centered optical instrument may be interpreted as defining a two-parameter transformation which generates a three-parameter group of Lie's first type. It is shown that all invariants of this group are functions of the one which occurs in a known formula due to Lagrange and Helmholtz.

13. In his second paper Dr. Lunn points out that the rectilinear oscillation of a particle is determined as simple harmonic by the condition that the dissipation of energy under a resistance proportional to the velocity shall be a minimum when the period and mean square displacement are assigned. The conjugate points bounding the interval of true minimum are determined.

14. Starting with Maxwell's definitions of volume and surface density of charge, and of density of energy, in terms of electric displacement and strength of field, Dr. Lunn gives in this paper a deduction of the conditions of electrostatic equilibrium without assuming the existence of the potential function, which on the contrary appears as a lagrangian indeterminate multiplier.

15. Mr. Schweitzer's note refers to a remark of Enriques in the Encyclopädie der Mathematischen Wissenschaften, volume III, 1, page 34. For the definition of euclidean geometry on the basis of the descriptive-projective geometry, we distinguish, in the well-known manner, between the proper and improper points of the space and agree that all improper points shall lie on the unique improper plane. Every elliptic polar system Σ_0 in this improper plane gives rise to a euclidean geometry $G(\Sigma_0)$.

Thus by means of the class of elliptic polar systems $\{\Sigma_0\}$ in the improper plane is defined a unique class of euclidean geometries $\{G(\Sigma_0)\}$. The members of this latter class are, however, given by the geometry function $G(\Sigma)$ of a variable elliptic polar system Σ in the improper plane. This geometry function is then uniquely defined and consists of a complete set of euclidean propositions in which the congruence propositions appear as implications, having as terms propositional functions of the variable Σ. (Compare Russell, The Principles of Mathematics, pages 13, 82.)

16. As formal types of relational, class, and operational propositions Mr. Schweitzer takes, aRb, $a\epsilon C(b)$, $O_b(a)$. These propositions may be considered as expressing, respectively, "a possesses the relation R with reference to b," "a is in the class C with reference to b," "the operation O with reference to b affects a." It is assumed that aRb implies $a\epsilon C(b)$ implies $O_b(a)$. The following analysis is studied:

$$aRb = aR + b = a + Rb,$$

$$a\epsilon C(b) = aC + b = a + Cb,$$

$$O_b(a) = bO + a = b + Oa,$$

where $aR \neq Ra$, etc., and the symbol of composition "$+$" is an indefinable which implies no ordering of the terms.

17. In his previous papers on the general projective theory of surfaces, Professor Wilczynski assumed that the surface was referred to its asymptotic curves. The object of the present memoir is to find the general expressions for the invariants and covariants if the parameters to which the surface is referred are arbitrary. In this way it becomes possible to apply the results of the author, as developed in his earlier papers, to a given surface without first determining its asymptotic curves. The paper will be offered for publication in the *Transactions*.

18. Dr. Birkhoff effects a classification of irregular integrals by proving that all such integrals are linearly expressible in terms of certain canonical irregular integrals; the coefficients in the linear form are analytic at the singular point $x = a$ in question. These canonical irregular integrals are of the form $x^\kappa\phi(x)$, where $\phi(x)$ is analytic except at $x = a$, and are fully

characterized by the specification of a certain finite set of constants. By means of a method wherein certain integral equations are used, it is found possible to make a direct study of the general irregular integrals whose characteristic properties depend on the aforesaid constants. With the aid of these results it is finally proved that with every irregular integral are associated certain functional equations. Such a functional equation for e^x is $e^{x+c} = e^c \cdot e^x$.

19. The differential calculus furnishes an easy method of finding the general tangent to any curve whose equation is in cartesian coordinates. In this paper Professor Carmichael obtains the same formula for algebraic curves without the aid of the calculus, and extends this result so as to apply to a large number of cases in which the curve is transcendental. The reasoning is of such elementary nature as to be readily understood by a student in his first course. And it is believed that this general formula may be introduced to the pupil's notice with profit. The paper will be offered for publication in the *American Mathematical Monthly.*

20. In his second paper Professor Carmichael points out the form of the cartesian equation of plane algebraic curves symmetrical with respect to each of two rectangular axes, and classifies such curves of the fourth degree, the grouping in classes being determined by certain geometric properties common to those of each class. The paper will be offered for publication in the *American Mathematical Monthly.*

21. In a recent article in the *Transactions of the American Mathematical Society,* Professor Kellogg found it necessary to borrow two facts from the geometry of continuously turning curves; his note presents the proofs of these facts. It will be published in the *American Mathematical Monthly.*

22. In his paper in the *Transactions,* volume 6 (1905), page 504, Professor Loewy transforms any linear group G into a group the matrix of whose general substitution is a compound matrix with square matrices A_{11}, A_{22}, $\cdots$, $A_{\mu\mu}$ in the main diagonal, and with zero matrices to the right of the diagonal; further, the totality of matrices A_{11} form a (maximal) completely reducible group, the A_{22} a completely reducible group, etc. This normalization may be accomplished by a transfor-

mation within a given domain Ω containing the coefficients of the substitutions of G. The groups A_{11}, A_{22}, $\cdots$ completely reducible in Ω, are uniquely determined by G and Ω, when equivalent (conjugate) groups are considered to be identical. The paper by Dr. Schur shows that, when equivalent groups are considered to be not distinct, the groups A_{11}, A_{22}, $\cdots$ are entirely independent of the choice of the domain Ω, so that, if A'_{11}, A'_{22}, $\cdots$, $A'_{\mu'\mu'}$ are the analogous groups for a new domain Ω' likewise containing the coefficients of the substitutions of G, then $\mu' = \mu$ and A'_{ii} is equivalent to A_{ii}. Of the various additional theorems established by Dr. Schur, mention may be made of the following: Every linear group G irreducible in a domain Ω is completely reducible in the domain Z of all numbers; if the most general matrix commutative with G has exactly r distinct characteristic roots, G decomposes in Z into r irreducible groups F, F_1, $\cdots$ of equal degree; these r groups may be represented in r conjugate algebraic domains $\Omega(\rho_i)$. If exactly l of these r groups are non-equivalent, then r/l is an integer which divides the degree of the group F. The paper will be offered for publication in the *Transactions*.

23. The character of the solutions of homogeneous linear differential equations with periodic coefficients (period 2π) is well known from the writings of Floquet, Callandreau, and others. The purpose of Mr. MacMillan's paper is to exhibit similarly the character of the solutions when the equations are linear with periodic coefficients but non-homogeneous. It is found that, when the characteristic exponents are all distinct and none of them congruent to zero (mod $\sqrt{-1}$) and the non-homogeneous terms have the period 2π, the particular solutions are periodic with the period 2π. If one of the characteristic exponents is congruent to zero (mod $\sqrt{-1}$), then in general the particular solution will contain terms involving the independent variable as a linear factor, that is to say the solution is not periodic.

If the non-homogeneous terms of the differential equations contain terms of the form $\sin (i + \lambda)t$ or $\cos (i + \lambda)t$ (where i is an integer, and λ is any real number) then the particular solution depending upon these terms is trigonometric, provided the characteristic exponents are all distinct and not congruent to zero nor equal to $\lambda\sqrt{-1}$. When these conditions are not fulfilled the solution in general is not periodic. The appropriate discussion for these cases is given.

24. In the *Mathematische Annalen* volume 12, page 376, Grassmann has defined the middle product of two vectors as a "formal" sum of their outer and inner products, namely, $AB = [AB] + \lambda[A/B]$, the product being associative if $\lambda = -1$ and non-associative if $\lambda = +1$. Mr. Schweitzer observes that the quaternion and its extension occur as operators in the grassmannian vector space of 2^n dimensions $(n = 1, 2, 3, \cdots)$. For example, for $n = 2$, if

$$X = x_1E_1 + x_2E_2 + x_3E_3 + x_4E_4,$$

the vectors $i'x$, $j'x$, $k'x$ are suitably defined and the vectors $i'/\lambda' \cdot X$, $j'/\lambda'' \cdot X$, $k'/\lambda''' \cdot X$, where λ', λ'', λ''' are constants different from zero, are determined by the specification that the outer product $X \cdot i'X \cdot j'X \cdot k'X$ shall be identically equal to $\rho(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2 E_1E_2E_3E_4$, where ρ is a constant. These vectors may also be determined by a simple tactical construction of a square matrix of 2^n elements. It is found then that as operators on the vector X,

$$\left(\frac{i'}{\lambda'}\right)^2 = \left(\frac{j'}{\lambda''}\right)^2 = \left(\frac{k'}{\lambda'''}\right)^2 = -1, \quad \frac{i'}{\lambda'} \cdot \frac{j'}{\lambda''} = -\epsilon\frac{k'}{\lambda'''},$$

$$\frac{i'}{\lambda'} \cdot \frac{j'}{\lambda''} \cdot \frac{k'}{\lambda'''} = \epsilon,$$

where $\epsilon^2 = 1$. The quaternion Q is defined as an operator $Q(X', X)$ which transforms a grassmannian vector X into the vector X'. For the fundamental unit vectors E_1, E_2, E_3, E_4 the following relations are valid: (1) If $E_{r_1}E_{r_2}$ is the grassmannian complement of $E_{r_3}E_{r_4}$, then $Q(E_{r_1}E_{r_2}) = \epsilon Q(E_{r_3}E_{r_4})$. (2) If E_{r_1}, E_{r_2}, E_{r_3} are any three unit vectors, then

$$Q(E_{r_1}E_{r_2}) \cdot Q(E_{r_2}E_{r_3}) = Q(E_{r_1}E_{r_3}).$$

(3) If $E_{r_1} \neq E_{r_2}$, then $Q(E_{r_1}E_{r_2}) = -Q(E_{r_2}E_{r_1})$. The generalization of these relations is discussed.

Thus, from the operative viewpoint, the quaternion appears in a real and homogeneous manner; also what may be called the operative sum and product of two grassmannian vectors in 2^n space is readily defined.

H. E. SLAUGHT,
Secretary of the Section.