Now K and J, increased by unity, give (apart from a multiple of 3) the number of sets of values for which a cubic form (with the coefficients not all zero) vanishes in the $GF[3]$ and the $GF[3^2]$, respectively. We find that *

$$J = K + \Delta^2 - \Delta \quad (\Delta = \text{discriminant}),$$
$$K^2 + K = J^2 + J.$$

But K is not a rational function of J (in view of the first and second forms below), nor J a rational function of K (in view of the second and third forms):

Form.	K	J	Δ
$x^3 - xy^2 + y^3$	-1	-1	1
$x^3 + xy^2$	0	-1	-1
x^3	0	0	0
$x^2y + xy^2$	-1	-1	1
x^2y	1	1	0
Vanishing	0	0	0

Every cubic can be transformed modulo 3 into one of those given in the table (*Transactions*, l. c., page 232).

THE UNIVERSITY OF CHICAGO,
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NOTE ON JACOBI'S EQUATION IN THE CALCULUS OF VARIATIONS.

BY PROFESSOR MAX MASON.

(Read before the American Mathematical Society, February 29, 1908.)

IN Weierstrass's theory of the calculus of variations † it is shown that the determinant

$$\omega = \frac{\partial y}{\partial t}\frac{\partial x}{\partial a} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial a}$$

formed from the equations $x = x(t, a)$, $y = y(t, a)$ of a family of extremals of the integral

* If we employ the invariant $P = \Delta + 1 - K$ (l. c., p. 211), we have
$$J = K^2 + K + P - 1.$$

† See for example Bolza, Lectures on the calculus of variations, Chicago, 1904.

$$J = \int F(x, y, x', y')dt$$

is a solution of Jacobi's equation

$$(\omega' F_1)' - \omega F_2 = 0.$$

This result, which is of fundamental importance in the theory, is obtained by differentiating the Euler equations of the extremals

$$F_x - \frac{d}{dt} F_{x'} = 0, \quad F_y - \frac{d}{dt} F_{y'} = 0$$

with respect to the parameter a, a method which involves considerable reckoning and the introduction of two sets of functions L, M, N; L_1, M_1, N_1, which serve to define F_2.

It is the object of this note to derive the result above stated directly from the single equation of the extremals

$$(1) \qquad T \equiv F_{xy'} - F_{yx'} + F_1(x'y'' - y'x'') = 0,$$

which is equivalent to the pair of dependent Euler equations. The introduction of successive sets of auxiliary functions to define F_2 is in this way avoided, and an explicit form for F_2 is obtained.

Write for abbreviation $\partial x/\partial a = \xi$, $\partial y/\partial a = \eta$, and denote differentiation with respect to t by accents. Then

$$\omega = y'\xi - x'\eta, \quad \omega' = y''\xi - x''\eta + y'\xi' - x'\eta',$$
$$\omega'' = y'''\xi - x'''\eta + 2(y''\xi' - x''\eta') + y'\xi'' - x'\eta''.$$

If equation (1) be differentiated with respect to a, and the quantity $[y'''\xi - x'''\eta + 3(y''\xi' - x''\eta')]F_1$ be subtracted and added in the result, the following equation is obtained:

$$(2) \qquad \begin{aligned} - \omega'' F_1 &+ \xi'[3y''F_1 + (x'y'' - y'x'')F_{1x'} - x'y'F_{1x} - y'^2 F_{1y}] \\ &+ \eta'[- 3x''F_1 + (x'y'' - y'x'')F_{1y'} + x'^2 F_{1x} + x'y'F_{1y}] \\ &+ \xi[y'''F_1 + (x'y'' - y'x'')F_{1x} + F_{xxy'} - F_{xyx'}] \\ &+ \eta[- x'''F_1 + (x'y'' - y'x'')F_{1y} + F_{xyy'} - F_{yyx'}] = 0. \end{aligned}$$

Since

$$F_1' = x''F_{1x'} + y''F_{1y'} + x'F_{1x} + y'F_{1y},$$

the coefficients of ξ' and η' are equal to

$$y''(3F_1 + x'F_{1x'} + y'F_{1y'}) - y'F_1',$$
$$- x''(3F_1 + x'F_{1x'} + y'F_{1y'}) + x'F_1',$$

respectively. Now it may be shown from the homogeneity property of F that

(3) $$3F_1 + x'F_{1x'} + y'F_{1y'} = 0.$$

In fact, on differentiating the identity

(4) $$x'F_{x'} + y'F_{y'} = F$$

twice with respect to x', the equation

$$F_{x'x'} + x'F_{x'x'x'} + y'F_{x'x'y'} = 0$$

is obtained. If the second derivatives be expressed in terms of F_1, this equation becomes

$$y'^2(3F_1 + x'F_{1x'} + y'F_{1y'}) = 0.$$

A similar equation, where the factor y'^2 is replaced by x'^2, is obtained by differentiating (4) with respect to y'. Since x' and y' are not simultaneously zero, equation (3) must hold. The coefficients of ξ' and η' in equation (2) are therefore $-y'F'$ and $x'F_1'$ respectively. After adding and subtracting the expression $(y''\xi - x''\eta)F_1'$, equation (2) takes the form

(5) $$-(\omega'F_1)' + P\xi + Q\eta = 0,$$

where

(6) $$P = y'''F_1 + (x'y'' - y'x'')F_{1x} + F_{y'xx} - F_{x'xy} + y''F_1',$$
$$Q = -x'''F_1 + (x'y'' - y'x'')F_{1y} + F_{y'xy} - F_{x'yy} - x''F_1'.$$

Now

$$x'P + y'Q = \frac{d}{dt}T = 0,$$

so that there exists a function F_2 such that

(7) $$P = y'F_2, \quad Q = -x'F_2.$$

Therefore, after changing the signs in equation (5) the desired equation

$$(\omega'F_1)' - \omega F_2 = 0$$

is obtained.

The function F_2 determined by equations (6) and (7) may be found explicitly from the equation

$$(x'^2 + y'^2)F_2 = y'P - x'Q.$$

On expanding the second member and collecting terms, this equation becomes

$$(x'^2 + y'^2)F_2 = (x'x''' + y'y''')F_1 + (x'x'' + y'y'')F_1' \\ - F'_{xx'} - F'_{yy'} + x'(F_{x'xx} + F_{x'yy}) + y'(F_{y'xx} + F_{y'yy}).$$

Now on differentiating the identity

$$x'F_{x'} + y'F_{y'} = F$$

twice with respect to x or y, the equations

$$x'F_{x'xx} + y'F_{y'xx} = F_{xx}, \quad x'F_{x'yy} + y'F_{y'yy} = F_{yy}$$

are obtained, so that F_2 is given by the equation

$$(x'^2 + y'^2)F_2 = (x'x''' + y'y''')F_1 \\ + (x'x'' + y'y'')F_1' + F_{xx} - F'_{xx'} + F_{yy} - F'_{yy'}.$$

In case the parameter t is the length of arc, so that $x'^2 + y'^2 \equiv 1$, the function F_2 has the simpler form

$$F_2 = F_{xx} - F'_{xx'} + F_{yy} - F'_{yy'} - (x''^2 + y''^2)F_1.$$

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ON THE DISTANCE FROM A POINT TO A SURFACE.

BY PROFESSOR E. R. HEDRICK.

(Read before the American Mathematical Society, September 5, 1907.)

THE discussion of the extrema of the distance from a point to a surface has been made the basis for the treatment of principal radii of curvature and for the classification of points on a surface by several writers.* In this connection it is interest-

* See, e. g., Goursat, Cours d'analyse, or English translation, no. 60 ; the statements there made are correct, the example here considered falling under the case $s^2 - rt = 0$. See also BULLETIN, vol. 13, no. 9, pp. 447, 448 ; the statements of this article differ in their spirit from those of the present article, and comparisons must be made with this understanding.