Thus the transpositions (12), (23), $\cdots$, (78) correspond to (4), (5), $\cdots$, (10), respectively. In the standard notations for abelian substitutions the latter are, respectively,

$$M_1 N_{31} Q_{31}, \quad M_3 N_{23} Q_{23}, \quad M_2, \quad M_3 Q_{31} N_{23} R_{13} Q_{23}, \quad M_1,$$
$$M_3 R_{31} Q_{31}, \quad M_2 N_{32} Q_{32}.$$

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# DOUBLE POINTS OF UNICURSAL CURVES.

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THE coordinates of a unicursal curve may be expressed as rational functions of a parameter. If we assume the curve to be of order $n$ and use non-homogeneous coordinates, we have

$$x = a(\lambda)/c(\lambda), \quad y = b(\lambda)/c(\lambda),$$

where $a$, $b$, $c$ are polynomials of order $n$ in the parameter $\lambda$. For the double points two values of the parameter give the same values of $x$ and $y$, and the usual method for their determination consists in finding pairs of values of $\lambda$ and $\mu$ that satisfy the equations

$$a(\lambda)/c(\lambda) = a(\mu)/c(\mu), \quad b(\lambda)/c(\lambda) = b(\mu)/c(\mu).$$

After elimination of $\mu$ from these equations and division ot the result by certain extraneous factors, an equation of order $(n - 1)(n - 2)$ in $\lambda$ is obtained, and the roots of this equation combine in pairs to give the parameters of the $\frac{1}{2}(n - 1)(n - 2)$ double points. The process of solution however involves the solution of an equation of order $(n - 1)(n - 2)$.

Suppose now that $a$, $b$, $c$ are polynomials in $\lambda$ with real coefficients, *i. e.*, suppose the curve real, and write $\lambda + i\mu$ for $\lambda$. Let $a$ be $A(\lambda, \mu^2) + i\mu A'(\lambda, \mu^2)$ and similarly for $b$ and $c$. It is clear that $\lambda + i\mu$ gives for $(x, y)$ the value

$$\left(\frac{A + i\mu A'}{C + i\mu C'}, \quad \frac{B + i\mu B'}{C + i\mu C'}\right)$$

and that $\lambda - i\mu$ gives

$$\left(\frac{A - i\mu A'}{C - i\mu C'}, \quad \frac{B - i\mu B'}{C - i\mu C'}\right).$$

These values are the same if

$$\frac{A'C - AC'}{C^2 + \mu^2 C'^2} = 0, \quad \frac{B'C - BC'}{C^2 + \mu^2 C'^2} = 0.$$

It is at once clear that the common values of $\lambda$, $\mu$ satisfying $A'C - AC' = 0$, $B'C - BC' = 0$ give the parameters $\lambda + i\mu$, $\lambda - i\mu$ of the double points, and in addition the values of $\lambda$, $\mu$ which make $C = 0$, $C' = 0$. Also, a real pair of values of $\lambda$, $\mu$ corresponds to a real isolated double point, whilst a real crunode is given by $\lambda$ real and $\mu$ purely imaginary. In either case $\mu^2$ is real and the two above equations are in $\lambda$ and $\mu^2$; hence each real pair of values of $\lambda$, $\mu^2$ gives a real double point and each imaginary pair an imaginary double point. For a crunode $\mu^2$ is negative, and for an isolated point it is positive.

The number of intersections of $C$ and $C'$ is $n(n-1)$, whilst the number of intersections of $A'C - AC'$ and $B'C - BC'$ is apparently $(2n-1)^2$. We shall now show that this number is in reality much less.

Suppose that

$$a(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots, \quad b(\lambda) = b_0\lambda^n + b_1\lambda^{n-1} + \cdots,$$
$$c(\lambda) = c_0\lambda^n + c_1\lambda^{n-1} + \cdots$$

and write

$$\lambda + i\mu = r(\cos\theta + i\sin\theta).$$

Then

$$\begin{aligned} A'C - AC' = \{&[a_0 r^n \sin n\theta + a_1 r^{n-1}\sin(n-1)\theta + \cdots] \\ &\times [c_0 r^n \cos n\theta + c_1 r^{n-1}\cos(n-1)\theta + \cdots] \\ &- [c_0 r^n \sin n\theta + c_1 r^{n-1}\sin(n-1)\theta + \cdots] \\ &\times [a_0 r^n \cos n\theta + a_1 r^{n-1}\cos(n-1)\theta + \cdots]\} \div r\sin\theta \end{aligned}$$

$$\begin{aligned} = (a_0c_1 - a_1c_0)r^{2n-2} &+ (a_0c_2 - a_2c_0)r^{2n-3}\frac{\sin 2\theta}{\sin\theta} \\ &+ \left[(a_0c_3 - a_3c_0)\frac{\sin 3\theta}{\sin\theta} + (a_1c_2 - a_2c_1)\right]r^{2n-4} \\ &+ \left[(a_0c_4 - a_4c_0)\frac{\sin 4\theta}{\sin\theta} + (a_1c_3 - a_3c_1)\frac{\sin 2\theta}{\sin\theta}\right]r^{2n-5} \\ &+ \cdots, \text{ etc.,} \end{aligned}$$

$$= (a_0c_1 - a_1c_0)(\lambda^2 + \mu^2)^{n-1} + (a_0c_2 - a_2c_0)2\lambda(\lambda^2 + \mu^2)^{n-2}$$
$$+ [(a_0c_3 - a_3c_0)(3\lambda^2 - \mu^2) + (a_1c_2 - a_2c_1)(\lambda^2 + \mu^2)](\lambda^2 + \mu^2)^{n-3}$$
$$+ [(a_0c_4 - a_4c_0)(4\lambda^3 - 4\lambda\mu^2) + (a_1c_3 - a_3c_1)2\lambda(\lambda^2 + \mu^2)](\lambda^2 + \mu^2)^{n-3}$$

plus terms of order lower than $2n - 5$.

Hence $A'C - AC'$ is of order $2n - 2$ and has a multiple point of order $n - 1$ at each of the circular points at infinity. Similarly $B'C - BC'$ has multiple points at the circular points. The number of other intersections of these two curves is therefore

$$(2n - 2)^2 - 2(n - 1)^2 = 2(n - 1)^2.$$

Of these $n(n - 1)$ are accounted for, and there remain

$$2(n - 1)^2 - 2(n - 1) = (n - 1)(n - 2).$$

Now the equations contain only even powers of $\mu$, and therefore if $\lambda$, $\mu$ be one intersection, $\lambda$, $-\mu$ is another. If $\mu^2$ be eliminated from them, and the extraneous polynomial in $\lambda$ arising from $C = 0$, $C' = 0$ be divided out, there remains an equation of order $\frac{1}{2}(n - 1)(n - 2)$ for $\lambda$. To each value of $\lambda$ corresponds one double point. The corresponding value of $\mu^2$ may in general be determined by elimination, and hence in general if $\lambda$ be real the double point is real.

As an example we consider the unicursal cubic

$$x = \frac{a_0\lambda^3 + a_1\lambda^2}{c_2\lambda + c_3}, \quad y = \frac{b_1\lambda^2 + b_2\lambda}{c_2\lambda + c_3}.$$

$$A'C - AC' = a_0c_2(\lambda^2 + \mu^2)2\lambda + a_0c_3(3\lambda^2 - \mu^2)$$
$$+ a_1c_2(\lambda^2 + \mu^2) + a_1c_3 2\lambda.$$

$$B'C - BC' = b_1c_2(\lambda^2 + \mu^2) + b_1c_3 2\lambda + b_2c_3.$$

The equations for $\lambda$ and $\mu$ give

(1) $\qquad b_1c_2(\lambda^2 + \mu^2) + b_1c_3 2\lambda + b_2c_3 = 0,$

(2) $\quad -4a_0b_1c_3(\lambda^2 + \mu^2) + a_1b_1c_2(\lambda^2 + \mu^2) + 2(a_1b_1c_3 - a_0b_2c_3)\lambda = 0.$

Hence

$$\frac{2b_1c_3\lambda + b_2c_3}{b_1c_2} = \frac{2(a_1b_1c_3 - a_0b_2c_3)\lambda}{(a_1c_2 - 4a_0c_3)b_1}$$

is the equation for $\lambda$, and the value of $\lambda$ from this equation, substituted in (1), gives the value of $\mu^2$.

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