ON A FINAL FORM OF THE THEOREM OF UNIFORM CONTINUITY.

BY PROFESSOR E. R. HEDRICK.

(Read before the American Mathematical Society, February 23, 1907.)

1. It is one of the fundamental theorems of analysis that a function which is continuous in a closed interval is uniformly continuous in that interval. A natural generalization to any closed assemblage is plausible and has been proved by Jordan;* again, an extension to a function of any number of variables is almost obvious. Another and a less obvious generalization has occurred to the writer, of which a special case has been stated by Baire.† In the case of a function of one variable this may be stated as follows:

Let $f(x)$ be a function defined on any assemblage (H) of values of x, and let (E) be a closed subassemblage of (H');‡ if the oscillation $\Omega(x)$ is defined with respect to the values of $f(x)$ on (H) in the usual manner§ and if $\Omega(x) \leqq k$ at each point of (E), then, corresponding to any positive number ϵ, there exists another positive number η, such that $|f(x)-f(\xi)| < k+\epsilon$ when ξ is any point of (E) and x is any point of (H) for which $|x-\xi| < \eta$.

The nature of the extension will appear directly; at this point attention is directed to the final character of this result, in its application to any function for values assumed on any assemblage.

2. The ordinary statements may be revised by use of the concept of oscillation, and it is desirable for what follows to do this. If a function $f(x)$ is defined on any assemblage (H) whose limiting points form another assemblage (H'), the values

---

*Jordan, Cours d'analyse, 2d ed., vol. 1, p. 48.

† Baire, Thesis: "Sur les fonctions de variables réelles," *Annali di Matem.*, 1899, p. 15; Baire, Leçons sur les fonctions discontinues, Paris, 1905; Borel, Leçons sur les fonctions de variables réelles, Paris, 1905, p. 27. Baire's statement makes (H) a continuum and $(E)=(H)$. See also W. H. Young, Theory of sets of points, Cambridge, 1906, p. 218.

‡ (H') is the assemblage of all the limiting points of (H), *i. e.*, the *first derived assemblage*. It is not necessary that (E) be part of (H'), but the real content of the theorem is the same if this restriction is made.

§ See § 2.

of $f(x)$ at points of (H) near a point h' of (H'), for example in the interval $(h' - \delta,\ h' + \delta)$, have lower and upper limits which may be denoted by

$$l[h', (H), \delta, f(x)] \quad \text{and} \quad L[h', (H), \delta, f(x)],$$

respectively, at least provided $f(x)$ is limited on (H).* Let $\omega[h', (H), \delta, f(x)] = L - l$ be called the oscillation of $f(x)$ on (H) in the interval $(h' - \delta, h' + \delta)$. When there is no ambiguity we shall write simply $L(h', \delta)$, $l(h', \delta)$, $\omega(h', \delta)$, respectively. Again, $\omega(h', \delta)$ is never negative, and it does not increase with δ; hence it has a lower limit $\Omega(h')$, which is defined at every point of (H'). $\Omega(h')$ is also the difference between the lower limit of $L(h', \delta)$ and the upper limit of $l(h', \delta)$, which we shall denote by $M(h')$ and $m(h')$ respectively.

It is evident that if (H) is an interval $a \leqq x \leqq b$, $f(x)$ is continuous at every point at which $\Omega(x) = 0$ and discontinuous at every point at which $\Omega(x) \neq 0$.

Hence the ordinary theorem may be stated as follows: *If (H) is an interval $a \leqq x \leqq b$ and if $\Omega(x) = 0$ for every x, then corresponding to every positive number ϵ there exists another positive number η independent of x, such that $\omega(x, \delta) < \epsilon$ whenever $\delta < \eta$.*

Jordan's form requires $\Omega(x) = 0$ for every x in a certain closed assemblage (E) of the interval, and shows that $\omega(x, \delta) < \epsilon$ for $\delta < \eta$ if x belongs to (E).

3. The proof of the theorem stated in § 1 is now easy. For if $\Omega(h') \leqq k$ for all points h' in (H') which belong to a closed subassemblage (E) of (H'), then corresponding to the given ϵ and for a fixed point e of (E) we can surely select $\eta(e)$ dependent on e and such that

$$\omega[e, (H), \delta, f(x)] < k + \epsilon \quad \text{whenever } 0 < \delta \leqq \eta(e),$$

since Ω is the lower limit of ω. Suppose now that $\eta(e)$ has a lower limit zero. Then there surely exists a monotone sequence of e's : $e_1, e_2, \cdots, e_n, \cdots$ such that

$$\lim_{i=\infty} \eta(e_i) = 0.$$

Let $\bar{e}$ be the single limiting point of this sequence; it lies in (E) since (E) is closed. Hence

$$\omega[\bar{e}, (H), \delta, f(x)] < k + \epsilon \quad \text{whenever } 0 < \delta \leqq \bar{\eta},$$

* Even this provision is unnecessary if one desires to introduce the ideal values $\pm\infty$.

where $\bar{\eta} = \eta(\bar{e})$. Let e_m be any e_i within the interval $(\bar{e} - \bar{\eta}/2, \bar{e} + \bar{\eta}/2)$; then surely we shall have

$$\omega[e_m, (H), \frac{\delta}{2}, f(x)] \leqq \omega[\bar{e}, (H), \delta, f(x)] < k + \epsilon$$
$$\text{whenever } \delta < \bar{\eta},$$

which is in contradiction with the assumption that $\eta(e_m)$ necessarily approaches zero as m becomes infinite.*

Let η denote the lower limit of $\eta(e)$ for points e in (E). Since $\eta \neq 0$ and since $\eta(e) \geqq \eta$ for every e in (E) it follows that

$$\omega[e, (H), \delta, f(x)] < k + \epsilon \quad \text{whenever } 0 < \delta < \eta,$$

where η is independent of e; and the theorem is proved: *If $f(x)$ is defined on (H), and if $\Omega(x) \leqq k$ for every x in a closed subassemblage (E) of (H'), then $\omega[e, (H), \delta, f(x)] < k + \epsilon$ whenever $0 < \delta < \eta$,* where *e is any point of (E), the order of choice being ϵ, η, e.*

This restatement is somewhat clearer from the standpoint of the oscillation.† It is evident that the theorem and the proof may be stated for a function of any number of variables without essential modification of either the notations or the arguments.

COLUMBIA, MO.,
February, 1907.

---

* This conclusion may also be reached, but hardly more simply, by applying an extension of Borel's theorem (Borel, l. c., p. 9) on the possibility of covering an interval by a finite number of intervals, if we first extend Borel's theorem to fit the case of intervals enclosing the points of any closed assemblage, as is obviously possible.—(Note, March 15.) My attention is called by Professor J. W. Young to the fact that this extension of Borel's theorem is stated specifically by Veblen, BULLETIN, vol. 10, p. 436; and by W. H. Young, Theory of sets of points, Cambridge, 1906, p. 41.

† The oscillation in an interval, $\omega(x, \delta)$ approaches its limits $\Omega(x)$ (the oscillation at a point), *uniformly* if $\Omega(x) = 0$. A mistaken idea may suggest itself that ω approaches its limit Ω uniformly in any case; this is not true, as is seen in the case of functions which are continuous at least once in every interval and also discontinuous at least once in every interval; the theorem stated is all the more remarkable on this account.