

THE FEBRUARY MEETING OF THE AMERICAN  
MATHEMATICAL SOCIETY.

THE one hundred and thirty-second regular meeting of the Society was held in New York City on Saturday, February 23, 1907. The attendance at the two sessions included the following thirty-three members :

Professor G. A. Bliss, Professor E. W. Brown, Dr. W. H. Bussey, Professor F. N. Cole, Dr. W. S. Dennett, Dr. G. B. Germann, Dr. A. M. Hiltebeitel, Professor Edward Kasner, Dr. G. H. Ling, Professor E. O. Lovett, Professor Max Mason, Mr. A. R. Maxson, Professor H. B. Mitchell, Dr. R. L. Moore, Mr. H. W. Reddick, Professor L. W. Reid, Dr. R. G. D. Richardson, Miss I. M. Schottenfels, Mr. L. P. Sicelof, Dr. Clara E. Smith, Professor P. F. Smith, Dr. R. P. Stephens, Dr. W. M. Strong, Professor J. H. Tanner, Professor H. D. Thompson, Dr. A. L. Underhill, Professor Oswald Veblen, Professor L. A. Wait, Mr. H. E. Webb, Professor H. S. White, Professor E. B. Wilson, Dr. Ruth G. Wood, Professor J. W. Young.

The President of the Society, Professor H. S. White, and Vice-President Professor P. F. Smith occupied the chair. The Council announced the election of the following persons to membership in the Society : Professor T. M. Focke, Case School of Applied Science, Cleveland, Ohio ; Dr. D. C. Gillespie, Cornell University ; Professor C. C. Grove, Hamilton College, Clinton, N. Y. ; Professor T. W. Palmer, University of Alabama ; Professor N. A. Pattillo, Randolph-Macon Woman's College ; Mr. F. D. Posey, University of Chicago ; Miss Gertrude Smith, Vassar College ; Dr. A. L. Underhill, Princeton University. Ten applications for admission to the Society were received.

After seven years' service as Treasurer, Dr. W. S. Dennett expressed the wish to be relieved of the duties of that office. The vacancy was filled by the appointment of Professor J. H. Tanner. Professor Maxime Bôcher was elected a member of the Editorial Committee of the *Transactions*, to succeed Professor E. H. Moore, who will retire, at the completion of the present volume of that journal, after eight years' service as editor in chief. Appropriate resolutions expressing grateful appreciation of the services of these two officers were adopted by the Society.

A standing committee of the Council, consisting of the Treasurer as chairman, Professor Brown, and Dr. Dennett, was appointed to have charge of the investment of the Society's life membership and surplus funds.

For the better regulation of the presentation of papers at the meetings of the Society, the rule was adopted that hereafter papers shall be read in the order and at the session assigned them on the printed programme. Papers whose reading is postponed at the request of the authors or in their absence will be read at the close of the session, if time permits, or at the close of the last session. Papers not announced on the printed programme, but subsequently accepted for presentation, may be read so far as time permits at the close of any session, after the printed list has been exhausted, or at the close of the last session.

To facilitate discussion, abstracts of papers will be included in the printed programme if furnished by the authors for that purpose at least three weeks in advance of the meeting. Such abstracts should be in the usual form, but should also be accompanied by standard and special references to the literature of the subject.

The following papers were read at this meeting:

(1) Professor R. D. CARMICHAEL: "On dividing an angle into parts having the ratio of any given straight lines."

(2) Professor R. D. CARMICHAEL: "A table of multiply perfect numbers."

(3) Professor G. A. MILLER: "The groups generated by three operators each of which is the product of the other two."

(4) Dr. R. P. STEPHENS: "On a quintic with three parallel tangents in any direction" (preliminary communication).

(5) Professor E. B. WILSON: "On the revolutions of a dark body about the sun."

(6) Mr. C. N. MOORE: "On the introduction of convergence factors into summable series and summable integrals."

(7) Professor G. A. BLISS: "The construction of a field of extremals about a given point."

(8) Dr. R. G. D. RICHARDSON: "Differentiation and integration of definite integrals."

(9) Professor E. R. HEDRICK: "On a final form of the theorem of uniform continuity."

(10) Professor R. D. CARMICHAEL: "On the classification of quartic curves possessing fourfold symmetry with respect to a point."

Mr. Moore's paper was communicated to the Society through Professor Bôcher. In the absence of the authors the papers of Professor Carmichael, Professor Miller, Mr. Moore, and Professor Hedrick were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In this paper Professor Carmichael considers the problem of dividing an angle into parts having the ratios of any given straight lines, and effects a solution by means of the locus of the polar equation  $\rho \sin \theta = \theta$ . A method of constructing the curve by continuous motion is also given. Once the curve is constructed, the division of the angle into the proper parts is a very simple matter.

2. In his paper on multiply perfect numbers Professor Carmichael exhibits a method for determining all the multiply perfect numbers up to 1,000,000,000. He gives a table of these numbers. In addition he gives another table which contains such other numbers as are known to him to be multiply perfect. It is interesting to note that this computation does not discover a multiply perfect odd number. It therefore still remains an unsettled question whether multiply perfect odd numbers exist.

3. Professor Miller proved that if three operators are such that the product of any two is equal to the third they generate either a dihedral group or the quaternion group. When the three defining equations do not admit the cyclic permutation of the operators, the latter generate a dihedral group, and every dihedral group can be generated by three such operators. When the equations admit such a cyclic permutation, the three operators in question generate one of the following four groups: identity, the group of order two, the four-group, or the quaternion group. Moreover, the operators may be so chosen as to satisfy the given conditions and generate any one of these four groups.

4. In an article, "On the pentadeltoid" in the *Transactions* for April, 1906, Dr. Stephens discussed a quintic whose parametric equation is

$$t^5 + \alpha \mu t^4 + x t^3 + \alpha y t^2 + \mu t + \alpha = 0,$$

to which can be drawn in any given direction only one tangent. A quintic of similar mechanical construction is

$$t^5 - xt^4 + \mu t^3 - \alpha \mu t^2 + \alpha y t - \alpha = 0,$$

which is a curve of class five and degree eight. To this curve, say  $K$ , in any direction may be drawn three parallel tangents, the centroid  $C$  of whose points of tangency is independent of the given direction and is defined as the center of  $K$ . In general there are five cusps, but these may decrease by twos to one while the curve changes in shape from the five-cusped hypocycloid as one limit to the cardioid as the other. Inscribed in each curve  $K$  is a concentric ellipse which touches  $K$  in five points. The orthoptic curve is a limaçon which touches  $K$  in five points.

Corresponding theorems may be proved for the general case where the equation is

$$t^n + xt^{n-1} + a_1 t^{n-2} + a_2 t^{n-3} + \dots + b_2 t^3 + b_1 t^2 + yt + 1 = 0;$$

for instance, this is a curve to which  $n - 2$  parallel tangents may be drawn in any given direction.

5. In this paper Professor Wilson takes up mathematically the problem discussed by Poynting, in a lecture before the Royal Society recently printed in *Nature*. The differential equations of motion of a particle under the action of the sun's radiation as well as gravitational attraction are such that the equation of the orbit may readily be expressed in terms of Bessel functions. The discussion of the formulas brings out simple approximate expressions for the decrease of the radius vector, the diminution of the eccentricity, and the rotation of the line of apsides of the orbit. It also appears that the total number of revolutions which a particle can make about the sun in an infinite time is finite even when the particle starts at an infinite distance.

6. The most important results of Mr. Moore's paper are contained in the following two theorems:

I. If the series  $u_0 + u_1 + u_2 + \dots$  is summable in the sense of Frobenius (*Crelle*, volume 89) and has the value  $S$ , then the series  $u_0 + u_1 f_1(\alpha) + u_2 f_2(\alpha) + \dots$  will be absolutely convergent for all positive values of  $\alpha$ , and its value  $F(\alpha)$  will be continuous

for such values and will approach  $S$  as its limit when  $\alpha = +0$ , provided the convergence factors  $f_n(\alpha)$  satisfy for all positive, integral values of  $n$  the following conditions: (a)  $f_n(\alpha)$  is continuous ( $\alpha \geq 0$ ); (b)  $|f_n(\alpha)| < N/n^{2+\rho}\alpha^{2+\rho}$  ( $\alpha > 0$ ); (c)  $f_n(0) = 1$ ; (d)  $f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha) \geq 0$  ( $0 \leq n\alpha \leq c$ ); (e)  $|f_n(\alpha) - 2f_{n+1}(\alpha) + f_{n+2}(\alpha)| < L/n^{2+\rho}\alpha^\rho$  ( $\alpha > 0$ ); where  $N, \rho, c$  and  $L$  are positive constants.

II. If the function  $f(x)$  is uniformly continuous for all values of  $x \geq a > 0$ , and if the integral

$$\int_a^\infty f(x)dx$$

is summable and has the value  $S$ , that is, if

$$\lim_{x=\infty} \left( \frac{1}{x} \int_a^x \int_a^a f(\beta)d\beta d\alpha \right) = S,$$

then the integral

$$\int_a^\infty f(x)\phi(\alpha, x)dx$$

will be absolutely convergent for all positive values of  $\alpha$ , and its value  $F(\alpha)$  will be continuous for such values and will approach  $S$  as its limit when  $\alpha = +0$ , provided the convergence factor  $\phi(\alpha, x)$  satisfies the following conditions: (a)  $\phi(\alpha, x)$  is continuous ( $x \geq a, \alpha \geq 0$ ); (b)  $\phi_x''(\alpha, x)$  exists and is continuous ( $x \geq a, \alpha \geq 0$ ); (c)  $|\phi(\alpha, x)| < N/x^{2+\rho}\alpha^{2+\rho}$  ( $x \geq a, \alpha > 0$ ); (d)  $\phi(0, x) = 1$  ( $x \geq a$ ); (e)  $\phi_x''(\alpha, x) \geq 0$  ( $0 \leq x\alpha \leq c$ ); (f)  $|\phi_x''(\alpha, x)| < L/x^{2+\rho}\alpha^\rho$  ( $x \geq a, \alpha > 0$ ); where  $N, \rho, c$  and  $L$  are positive constants.

Other theorems are proved for the case in which the original series or integral is convergent.

7. Professor Bliss's paper appears in full in the present number of the BULLETIN.

8. On the basis of the results obtained in former papers for the equality of the double integral and the iterated integral, Dr. Richardson is able to derive general theorems for the inversion of the order of integration. If one of the integrals

$$\int_X dx \int_Y |f| dy, \quad \int_Y dy' \int_X |f| dx$$

is finite, then

$$\int_X dx \int_Y f dy = \int_Y dy \int_X f dx.$$

This is true if certain restrictions are put on the arrangement of the points of discontinuity, these restrictions being necessary to insure integrability. These results are applied to a discussion of the continuity and differentiation of a definite integral depending on a parameter.

9. Professor Hedrick presented a final generalization of the theorem that a function continuous in a closed interval is uniformly continuous in that interval. The generalization is stated in terms of the oscillation  $\omega(x, \delta)$  in an interval  $(x - \delta, x + \delta)$  and the oscillation  $\Omega(x)$  at a point. If then  $\Omega(x) \leq \lambda$  for values of  $x$  in a closed assemblage ( $E$ ) for the values of  $f(x)$  which correspond to values of  $x$  in any assemblage ( $H$ ), then  $\omega(x, \delta) < \lambda + \epsilon$  whenever  $\delta < \eta$ , for all  $x$  in ( $E$ ), the order of choice being  $\epsilon, \eta, x$ .

10. Professor Carmichael determines the simplest general forms of the equation of quartic curves having fourfold symmetry with respect to a point, the point being taken as the origin. He shows that there are the following five classes which may be combined into two groups, the first containing I, IV, and V, and the second II and III:

$$\text{I. } x^4 + y^4 + c_1xy(x^2 - y^2) + c_2x^2y^2 + c_3(x^2 + y^2) + c_4 = 0.$$

$$\text{II. } x^4 - y^4 + r_1xy(x^2 + y^2) + r_2(x^2 - y^2) + r_3xy = 0.$$

$$\text{III. } x^3y + xy^3 + p_1(x^2 - y^2) + p_2xy = 0.$$

$$\text{IV. } x^3y - xy^3 + q_1x^2y^2 + q_2(x^2 + y^2) + q_3 = 0.$$

$$\text{V. } x^2y^2 + s_1(x^2 + y^2) + s_2 = 0.$$

F. N. COLE,  
*Secretary.*