ture, $\kappa \cos \theta$ is κ_g, the geodesic curvature of the curve. Thus *

$$(1) \qquad \delta \int_{t_0}^{t_1} \sqrt{x'^2 + y'^2 + z'^2} dt = \delta\tau_1 - \delta\tau_0 - \int_{s_0}^{s_1} \delta\nu \cdot \kappa_g \cdot ds$$

if $\delta\tau_1$, $\delta\tau_0$ are the components of δx, δy, δz in the directions of the tangents at the ends of the curve.

The above process gives a very simple treatment of your investigation in the *Mathematische Annalen*, volume 57, page 48. For, with a closed curve, the variation of the area is evidently $\int \delta\nu \cdot ds$; and so by Euler's rule we get

$$\int \delta\nu (1 - \lambda \cdot \kappa_g) ds = 0,$$

or

$$\kappa_g = 1/\lambda.$$

Incidentally, it would appear that the calculus of variations is almost the easiest way of finding the value of κ_g in terms of Gauss's coördinates E, F, G, u, v, by comparing the formula (1) above with the result of varying

$$\int_{t_0}^{t_1} \sqrt{Eu'^2 + 2Fu'v' + Gv'^2}\, dt.$$

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QUATERNIONS.

A Manual of Quaternions. By CHARLES JASPER JOLY. London, Macmillan and Co., 1905. 8vo. xxvii + 320 pages.

A QUATERNION, including a scalar and a vector as particular quaternions, is a number. This incontestable fact has not always been remembered in the discussions relating to quaternions, vector analysis, and related topics. The familiar history of the discovery and development of quaternions shows that Hamilton always kept this fact in mind, and never permitted the double use of the word *vector* to indicate a particular (right) quaternion, and its geometric representation, to confuse his

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* The formula (1) is given in Darboux's Théorie des Surfaces, vol. 3, p. 105, but is there derived by entirely different methods.

ideas. He extended our number-system into the broader domain of linear associative algebra, and gave us the only extension in which division remains possible, and the first known form of the long roll of quaternionic algebras. The dust raised by contending parties over the question of what system of algebraic symbols and methods would be most useful in applied mathematics has sometimes obscured the fundamental facts in the case of quaternions. In the opinion of the reviewer this is due to the efforts to establish the practical utility of various algorithms, or to establish priority claims, at the expense of scientific development. But all this aside, the fact remains that real quaternions is the next extension of algebra beyond ordinary complex numbers. The field which will be developed in the coming years, as the field of the theory of functions of a complex variable has been in the century just closed, is that of the functions of quaternions.

The mantle of Hamilton passed to Tait, and from him to Joly. Thus it happens, as the author remarks in his preface: "By a curious series of events one of Hamilton's successors at the Observatory of Trinity College has felt himself obliged to endeavor to carry out to the best of his ability Hamilton's original intention. And on the centenary of Hamilton's birth a Manual of Quaternions is offered to the mathematical world." And (be it said with all due respect to the late Professor Tait) the volume seems to be the long-expected one, which will open successfully this, to many, forbidding land. It is essentially a manual, containing in small compass a wealth of results not excelled in range even by Hamilton's or Tait's Elements. For many rather inaccessible memoirs have been drawn upon for processes, formulas, applications, proofs, and suggestions not to be found in either Hamilton or Tait. The whole is arranged for ready reference, with an analytic table of contents. It might seem that even the exercises should have been well indexed, as they are full of new matter. A point for commendation is the absence of controversial or metaphysical discussion. If one might suggest a chapter which would not have occupied many pages, yet would have been valuable, it would be one on the analytic theory; including the characteristic equation, the roots, axes, the analytic function, number theory, group theory, with extensions to nonions in the chapters on the linear vector function. A development of this kind directly from the quadratic equation is feasible and teachable, and prevents over-emphasis of the geometric interpretation.

As a matter of expediency, in presentation, Professor Joly begins with the vector, as a geometric entity. He defines $S.\alpha\beta$ and $V.\alpha\beta$, then the product $\alpha\beta$ is defined as the sum of $S.\alpha\beta$ and $V.\alpha\beta$. This product is a *quaternion*. The distributive, associative, and non-commutative laws follow at once. Much fruitless discussion on what a *product* is, has simply made plain that a product is what one chooses to make it. There is certainly no inherent reason why $\alpha\beta$ should be the sum of a scalar and a vector any more than there is why it should be a parallelogram. It might be neither. Those who prefer to call any bilinear combination of the coördinates of α and β a product, in particular $-S.\alpha\beta$ the direct product, $V.\alpha\beta$ the skew product, abandon what Hamilton always kept in mind as an algebraist — the flexible associative law. (For $S.\alpha S\beta\gamma$ vanishes completely and $V.\alpha V\beta\gamma$ is not equal to $V.(V\alpha\beta)\gamma$.) If the associative law is to be retained, we must accept quaternions as it is (or in some "equivalent" form), and $\alpha\beta$ is *the* product. Professor Joly wisely does not even refer to these discussions. He proceeds to define the symbols K, T, V, and develops the usual formulas. This development includes the rotator $q(\ )q^{-1}$, and some mention of imaginary and complex quaternions.

Chapters V and VI apply quaternions to the straight line, the plane, and the sphere. Many developments given by Hamilton appear in brief. The complex, the congruence, and the regulus are noticed; homogeneous coördinates appear in various forms; inversion is sketched, and the common theorems of solid geometry appear.

Chapter VII defines differentiation and the famous operator ∇. The definition given for this operator is especially useful:

$$\nabla = -\frac{V.d'\rho d''\rho\,.\,d(\ ) + V.d''\rho d\rho\,.\,d'(\ ) + V.d\rho d'\rho\,.\,d''(\ )}{S.d\rho d'\rho d''\rho},$$

in which $d\rho$, $d'\rho$, and $d''\rho$ are any three non-coplanar differentials of ρ, and d, d', d'' are corresponding variations of the operand in these three directions. From this the "canonical" form

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

is deduced. The "integral" definition of ∇ is also given:

$$\nabla.F\rho = \text{Lim}\,.\frac{1}{v}\int d\nu F\rho,$$

where dv is an outwardly directed element of vector area of any small closed surface surrounding the extremity of ρ, v is the volume enclosed by the surface, and $F\rho$ is any quaternion function. The geometric meaning and some mechanical meanings of ∇ are evident. Taylor's series and stationary values of a function are discussed. The examples include problems on roulettes and relative motion.

Chapter VIII treats of the linear vector function. Well-known formulas are developed, but in addition we find a consideration of the products of two functions, with geometric applications of their invariants and covariants. Professor Joly might have included even more of his own investigations with profit.*

The gaussian operator, $cq(\ )q^{-1}$, is introduced in an example. It may be remarked that neither these operators nor any linear vector functions can take the place of the quaternion, even though their multiplication table be isomorphic with that of quaternions. In fact, the exact statement of Cayley's identification of quaternions and matrices of order two, is that every matrix is isomorphic with a quadrate algebra. Another example suggests Hamilton's icosian calculus, and might have been used to introduce groups of quaternions.

Chapters IX and X discuss quadric surfaces, and curves and surfaces in general. The transformations and ideas of Hamilton are closely followed. Much differential geometry appears. The emanant line is used, and vector curvature, vector torsion, vector twist are discussed. Ruled surfaces, curvatures, geodesics, and families of surfaces follow. Many theorems appear in the examples.

Chapters XI, XII, and XIII treat of statics, finite displacements, and strain. The reduction of wrenches on a screw is amply explained, including compounding of wrenches on two and three screws, and resolution on six screws. Astatics is briefly developed. The notes in Volume 2 of Joly's Hamilton's Elements develop these subjects more extensively. A displacement is treated as a *twist about a screw*, velocity as a *twist-velocity*. The principles of the theory of the fixed and the moving axes follow. Homogeneous strain is developed along

* "Theory of linear vector functions," *Dublin Trans.* 30 (1894), pp. 597–647. "Scalar invariants of two linear vector functions," *Dublin Trans.* 30 (1895), pp. 709–728. "Quaternion invariants of linear vector functions and quaternion determinants," *Dublin Proc.* 4 (1896), pp. 1–16.

lines made familiar by Tait, and the closing sections touch upon non-homogeneous strain.

Chapters XIV and XV are on dynamics. The equations of motion of a particle under different forces are presented; as damped vibrations, central forces, constrained motion, and brachistochrones. The dynamics of a rigid body and the theory of screws follows at some length.

Chapter XVI is devoted to the applications of the differentiator ∇. The invariants of ϕ, where

$$\phi a = - Sa\nabla.\sigma,$$

are developed. Of these invariants

$$m'' = - S\nabla\sigma = \text{divergence}, \quad 2\epsilon = V\nabla\sigma = \text{curl},$$

and if $\sigma = iu + jv + kw$,

$$m = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \text{Jacobian}.$$

Integrations involving singularities are handled briefly, followed by various expressions useful in physics. These include ∇^{-1}, ∇^{-2}, spherical harmonics, expansion of functions, and various expressions for ∇ and ∇^2. The kinematics of a deformable system, including equations of continuity, flow along a curve, circulation for a closed curve, flux through a surface, irrotational and solenoidal vectors, have ample space. The equations of motion of a deformable system, stress function for viscous fluid and isotropic solid, dissipation function, elastic solid with plane waves in elastic solids, are skilfully handled. The Maxwell electromagnetic theory is given, and includes circuital laws, impressed electric and magnetic forces, and propagation of light in a crystalline medium. If any addition is to be desired to this excellent chapter, it might be a brief treatment of the electron theory.

Chapters XVII and XVIII apply quaternions to projective geometry and extend quaternions into alternate algebras of higher orders, applicable to space of n dimensions. The projective application is made by affixing to a point of mass w and vector ρ (origin arbitrary) the quaternion $q = w(1 + \rho)$. A unit point is then $Q = 1 + \rho$. The origin is the scalar point $\theta = 1$. A vector is the affix of the point at infinity in its direction. Following Hamilton's rather awkward notation, Professor Joly defines:

$$(a, b) = bSa - aSb, \quad [a, b] = V.VaVb,$$

$$(a, b, c) = S.VaVbVc,$$

$$[a, b, c] = (a, b, c) - [b, c]Sa - [c, a]Sb - [a, b]Sc,$$

$$(a, b, c, d) = S.a[b, c, d].$$

For unit points, we have

$$(A, B) = B - A = \beta - \alpha,$$

$$[A, B] = V.VAVB = V.\alpha\beta = V.\alpha(\beta - \alpha),$$

$$[A, B, C] = S\alpha\beta\gamma - V\beta\gamma - V\gamma\alpha - V\alpha\beta, \quad (A, B, C) = S.\alpha\beta\gamma,$$

$$(ABCD) = S\beta\gamma\delta - S\alpha\gamma\delta + S\alpha\beta\delta - S\alpha\beta\gamma.$$

The line ab is $[q, a, b] = 0$; the plane a, b, c, is $(q, a, b, c) = 0$. Other alternating functions are considered. The general linear transformation of points in space leading to a linear quaternion operator is treated at length. Projective geometry finds here a simple and natural statement. The theorems include reciprocity theorems, poles and polars, invariants of surfaces, the quaternion form of Aronhold's symbolic notation, quadrics under different relations, generalized confocals, general curves and surfaces, jacobians and hessians.

The last chapter presents the principles of a general algebra for hyperspace.* In this algebra we define

$$\alpha\beta = V_2\alpha\beta + V_0\alpha\beta, \quad \beta\alpha = - V_2\alpha\beta + V_0\alpha\beta.$$

$V_0\,\alpha\beta$ corresponds to the scalar of three dimensions, V_2 to the vector. The n units being $i_1, \cdots, i_n$ we have

$$i_s^2 = -1, \quad i_t^2 = -1, \quad i_s i_t + i_t i_s = 0, \quad s, t = 1, \cdots, n.$$

All products are associative and doubly distributive. Thus, if

* Cf. JOLY: "The associative algebra applicable to hyperspace," *Dublin Proc.* 21 (1898) [(3) vol. 5], pp. 73–123. "On the place of the Ausdehnungslehre in the general associative algebra of the quaternion type," *Dublin Proc.* 22 (1900) [(3) vol. 6], pp. 13–18.

$$\alpha = \Sigma x_1 i_1, \quad \beta = \Sigma y_1 i_1, \quad \gamma = \Sigma z_1 i_1,$$

$$\alpha\beta\gamma = V_3 .\, \alpha\beta\gamma + V_1\alpha\beta\gamma, \quad V_3\alpha\beta\gamma = \Sigma \mid x_1 y_2 z_3 \mid i_1 i_2 i_3,$$

$$V_1\alpha\beta\gamma = -\Sigma y_1 z_1 \Sigma x_1 i_1 + \Sigma x_1 z_1 \Sigma y_1 i_1 - \Sigma x_1 y_1 \Sigma z_1 i_1,$$

$$\alpha\beta\gamma\delta = V_4\alpha\beta\gamma\delta + V_2\alpha\beta\gamma\delta + V_0\alpha\beta\gamma\delta,$$

and so on for higher products. The different "products" of the Ausdehnungslehre appear at once in their proper places as partial products among all the partial products of one *associative* product. This algebra is of order $2n$, and, if $n = 2m$, is the product of m independent quaternion algebras, if $n = 2m + 1$, is the product of m independent quaternion algebras and the algebra of positives and negatives. Division is possible under much the same restrictions as in a Weierstrass commutative algebra. We also see that the multiple algebra of n-dimensional space is not necessarily a theory of matrices of order n.

The volume is what its author intended it to be, a handy manual for those who desire to learn quaternions and quaternion methods. The appearance of the book is pleasing, and to the reviewer it seems that the simple notation of Hamilton can scarcely be called improved when one views a page full of heavy type, artificial signs, and foreign alphabets. We believe a letter is superior to an arbitrary mark for indicating a process, and indices clearer than fonts of type and sets of alphabets. Professor Joly's text speaks for itself. The book appeals, finally, to the pure mathematician as well as to the physicist.

JAMES BYRNIE SHAW.

SHORTER NOTICES.

Geschichte der Mathematik in XVI. und XVII. Jahrhundert. By H. G. ZEUTHEN, Deutsche Ausgabe von RAPHAEL MEYER. *Abhandlungen zur Geschichte der mathematischen Wissenschaften,* XVII. Heft. Leipzig, B. G. Teubner, 1903. Pp. viii + 434. Price 16 Marks.

AT first thought it may seem strange that Professor Zeuthen should attempt to go over the same ground so recently covered by Cantor in the latter part of volume II and the first part of volume III of the latter's Vorlesungen. A little investigation