A SURVEY OF THE DEVELOPMENT OF GEOMETRIC METHODS.*

ADDRESS DELIVERED BEFORE THE SECTION OF GEOMETRY OF THE INTERNATIONAL CONGRESS OF ARTS AND SCIENCE, ST. LOUIS, SEPTEMBER 24, 1904.

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I.

To understand thoroughly the progress made in geometry during the century which has recently closed it is necessary to glance rapidly at the state of the mathematical sciences at the beginning of the nineteenth century. It is known that in the last period of his life, Lagrange, fatigued by those researches in analysis and mechanics which have assured to him an immortal renown, began to neglect mathematics for chemistry which according to him was becoming as easy as algebra, for physics, and for philosophic speculations. This state of mind of Lagrange is almost always found at certain times in the lives of the greatest scholars. Those new ideas which have come to them in the productive period of youth, and which they have introduced into the common domain of knowledge, have yielded to them all that can be expected of them; the man has fulfilled his task, and feels the need of turning the activities of his mind toward entirely new subjects. This need, it is necessary to recollect, began to manifest itself with especial force in Lagrange's time. For at that time the programme of the investigations opened to geometricians by the discovery of the infinitesimal calculus appeared to be nearly exhausted. A few differential equations more or less complicated to be integrated, a few chapters to be added to the integral calculus, and it seemed that the very limits of the science would be reached. Laplace was finishing the explanation of the system of the world and laying the foundation for molecular physics. New ways indeed were opening for the experimental sciences and were preparing for the astonishing development which those sciences were to receive during the century now closed. Am-

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père, Poisson, Fourier, and even Cauchy, the creator of the theory of imaginary quantities, directed their best efforts to the application of the analytic methods to mechanics and molecular physics; they seemed to believe that just outside of this new domain, which they were in such haste to scour, the limits of theoretical science were rigidly fixed.

Modern geometry — this we must claim as its title to distinction — arrived at the very end of the eighteenth century to contribute in large measure to the rejuvenation of the whole of mathematical science by offering a new and productive field for investigation and especially by proving to us, by means of brilliant successes, that general methods do not exhaust a science, but that even in the simplest subject there remains much to be done by an ingenious and inventive mind. The beautiful geometric methods of Huygens, Newton, and Clairaut had been forgotten or neglected. The brilliant ideas introduced by Desargues and Pascal had been left without development, and seemed to have fallen on a barren soil. Carnot, with his Essai sur les transversales and the Géométrie de position, and Monge especially, with the descriptive geometry which he created and his beautiful theories upon the generation of surfaces, appeared just at this time and welded together the chain which seemed broken. Thanks to these two scholars the ideas of Descartes and Fermat, the inventors of analytic geometry, took beside the infinitesimal calculus of Leibniz and Newton the place which had been supinely lost, but which should never have been abdicated. "With his geometry," says Lagrange, speaking of Monge, "this devil of a man will make himself immortal." And indeed descriptive geometry has not only enabled us to coordinate and perfect the processes of every art "where excellence and success in work and product are conditioned by precision of form," but also it has proved to be the graphic representation of a general and purely rational geometry whose fertility has been demonstrated by numerous and important investigations. Moreover, side by side with Monge's Géométrie Descriptive we must not forget to place his other masterpiece, the Application de l'analyse à la Géométrie; nor must we forget that to Monge is due not only the notion of lines of curvature, but also the elegant integration of the differential equation of these lines for the case of the ellipsoid, for which, it is said, Lagrange envied him. It is necessary to lay stress upon the character of the whole of Monge's work. He, the regenerator of modern geometry, pointed out from the beginning — though his successors may have forgotten it — that the alliance between geometry and analysis is useful and productive; and that perhaps this alliance is a condition for success to them both.

II.

Many geometers were molded in the school of Monge: Hachette, Brianchon, Chappuis, Binet, Lancret, Dupin, Malus, Gaultier de Tours, Poncelet, Chasles, etc.; among these Poncelet stands in the first rank. Neglecting everything in Monge's work that belongs to cartesian analysis or concerns infinitesimal geometry, he gave his efforts exclusively to the development of the purely geometric germs contained in the researches of his illustrious predecessor. Made prisoner by the Russians in 1813 at the crossing of the Dnieper and confined at Saratoff, Poncelet employed the leisure of captivity in demonstrating those principles which he developed in the Traité des propriétés projectives des figures, published in 1822, and in his great memoirs on reciprocal polars and harmonic means, which go back nearly to the same epoch. And thus Saratoff may be considered the birthplace of modern geometry. Rewelding the chain broken after Pascal and Desargues, Poncelet introduced both homology and reciprocal polars, thus emphasizing at the very beginning of the science those productive ideas on which the evolution of the science depended during the fifty years following.

The methods of Poncelet, presented in opposition to the analytic geometry, naturally were not received with favor by the French analysts. But the importance and novelty of these methods was such that they were not slow in stirring up in many directions the most profound researches. In discovering his principles Poncelet had been alone; but several geometers almost at the same time came on the scene to study these principles in all their phases, and to deduce the essential results implicitly contained therein.

Gergonne was brilliantly editing, at just this time, a magazine which to-day has an inestimable value for the history of geometry. The *Annales de Mathématiques*, published from 1810 to 1831 at Nîmes, was for more than fifteen years the one journal in the whole world devoted exclusively to mathematical research. Gergonne, who has left to us in many regards an excellent model for a director of a scientific journal, had the faults of the qualities which made him successful, in that he collaborated, often against their will, with the authors of the memoirs submitted to him, making alterations which sometimes made an author say either more or less than he intended or desired. Nevertheless, he was keenly struck with the originality and range of Poncelet's discoveries. Already some simple methods of transformation of figures were known; homology even had been employed in the plane, but without extending it to space — as Poncelet did — but especially without its power and productivity being recognized. Moreover all these were point transformations, that is to say, they made a point correspond to a point. In introducing reciprocal polars, Poncelet showed in the highest degree the genius of the inventor; for he gave the first example of a transformation in which to a point corresponds something else than a point. Any method of transformation enables us to increase the number of theorems, but that of reciprocal polars has the advantage of causing to correspond to a proposition another proposition of totally different aspect. This was something essentially new. To exhibit this advantage, Gergonne invented the plan, since so prevalent, of articles written in double column, with the correlative propositions side by side; and his was the idea of replacing the demonstrations of Poncelet, which required the intervention of a conic or surface of the second degree, by the famous principle of duality, the significance of which, at first a little vague, was adequately explained in the discussions on the subject between Gergonne, Poncelet, and Plücker.

Bobillier, Chasles, Steiner, Lamé, Sturm and many others were at this time, with Plücker and Poncelet, assiduous contributors to the *Annales de Mathématiques*. Gergonne, having become rector of the academy of Montpellier, in 1831 was compelled to discontinue the publication of his journal. But the success which it had attained, and the thirst for research to the development of which it had contributed, had commenced to bear fruit. Quetelet had just founded in Belgium the *Correspondance mathématique et physique*. As early as 1826 Crelle brought out at Berlin the first numbers of his celebrated journal, in which he published the memoirs of Abel, Jacobi, and Steiner. Soon there were to appear also a large number of separate works in which the principles of modern geometry were presented and developed in a masterly manner.

The first of these, in 1827, was Möbius's Barycentric Calculus, a truly original work, remarkable for the depth of the conceptions and the clearness and rigor of expression; then, in 1828, Plücker's Analytisch-geometrische Entwickelungen — the second part appearing in 1831 — shortly followed by the System der analytischen Geometrie published by the same author in 1835 at Berlin. In 1832 Steiner brought out at Berlin his great work: Systematische Entwickelung der Abhängigkeit geometrischer Gestalten von einander; and in the following year the Geometrische Constructionen ausgeführt mittelst der geraden Linie und eines festen Kreises, in which a proposition of Poncelet regarding the employment of a single circle for elementary geometric constructions is confirmed by the most elegant examples. Finally, in 1830, Chasles sent to the Brussels academy — which, happily inspired, had offered a prize for the study of the principles of modern geometry — his celebrated Aperçu historique sur l'origine et le développement des méthodes en Géométrie, followed by the Mémoire sur deux principes généraux de la Science: la dualité et l'homographie, which was not published until 1837.

Time is here lacking to give an account of each of these beautiful investigations and to comment worthily upon them. Moreover to what could such a study conduct us except to a new verification of the general laws in the development of science. When the time is ripe, when fundamental principles have been discovered and stated, nothing stops the progress of ideas; the same discoveries, or discoveries almost equivalent, are made almost at the same instant, and in divers places. Without attempting a discussion of this kind, which moreover might appear useless or irritating, nevertheless it is of consequence for us to bring out distinctly a fundamental difference in the tendencies of the great mathematicians who about 1830 had just given to geometry an impetus before unknown.

III.

Those who, like Chasles and Steiner, devoted their whole lives to inquiries in pure geometry, opposed to analysis that which they called synthesis; and adopting in the main rather than in detail the tendencies of Poncelet, they proposed to institute an independent theory, a rival to the cartesian analysis.

Poncelet could not content himself with the insufficient resources furnished by the method of projection; to reach the imaginary, he was compelled to contrive that famous principle of continuity which gave rise to the long discussions between himself and Cauchy. Suitably formulated, this principle is excellent and can render great service. But Poncelet erred in refusing to present it as a simple consequence of analysis ; and, on the other hand, Cauchy was in the wrong in not being willing to acknowledge that his objections, sound no doubt when applied to certain transcendental figures, were without force in the applications made by the author of the Traité des propriétés projectives. Whatever opinion one may form on the subject of such a discussion, it at least shows in the clearest manner that the geometric system of Poncelet rested on an analytic basis ; and we know moreover, by the unfortunate publication of the Saratoff notes, that it was by the aid of cartesian analysis that the principles which serve as the base of the Traité des propriétés projectives were first established.

Somewhat later than Poncelet — who, moreover, abandoned geometry for mechanics, where his work has had a preponderating influence — Chasles, for whom a chair in higher geometry in the Faculty of Science at Paris was created in 1847, endeavored to found an entirely independent and autonomous geometric theory. This he developed in two works of the greatest importance : in the Traité de Géométrie supérieure, which dates from 1852, and in the Traité des sections coniques, unfortunately unfinished, of which the first part alone appeared in 1865.

In the preface to the first of these works the author indicates very clearly the three fundamental points which allow to the new doctrine participation in the benefits of analysis, and which seem to him to mark a step forward in the cultivation of the science. These are :

1. The introduction of the principle of signs, which simplifies both statements and proofs, and gives to Carnot's method of transversals the full range of which it is capable ;

2. The introduction of the imaginary, which takes the place of the principle of continuity, and furnishes demonstrations as general as those of analytic geometry ;

3. The simultaneous demonstration of propositions which are correlative, that is to say, those which correspond in virtue of the principle of duality.

While homography and correlation were certainly studied by Chasles in his writings, nevertheless in his exposition he systematically set aside the use of the transformations of figures, which, as he thought, could not take the place of direct demonstrations because they masked the origin and the true nature of the properties obtained by means of them. There is some truth in this judgment, but the very progress of the science causes us to consider it as too severe. If it often happens that, employed without discernment, transformations uselessly multiply the number of theorems, it is not therefore necessary to ignore the fact that often also they aid us in the better understanding of the nature of the very propositions to which they have been applied. Is it not the use of Poncelet's projection which has led to the very fertile distinction between projective and metrical properties, which moreover has made us acquainted with the great importance of the anharmonic ratio, whose essential property is found as far back as Pappus, and the fundamental rôle of which began to appear only after fifteen centuries in the efforts of modern geometry?

The introduction of the principle of signs was not so new as Chasles thought when he wrote his Traité de Géométrie supérieure. Already Möbius in his Barycentric Calculus had answered a desideratum of Carnot, and employed signs in the broadest and most precise manner, defining for the first time the sign of a segment and even that of an area. Later he was successful in extending the use of signs to lengths which are not upon the same line and to angles which have not their vertices at the same point. Moreover, Grassmann, whose mind was so like that of Möbius, must necessarily have used the principle of signs in the definitions which form the foundation of his very original method of studying the properties of extension.

The second characteristic which Chasles assigns to his system of geometry is the use of the imaginary. Here his method was truly new and he was able to illustrate it by exceedingly interesting examples. We shall always admire the beautiful theory which he has left to us, of confocal surfaces of the second degree, where all the known properties and other new ones — as various as elegant — are derived from the general principle that they are inscribed in the same developable circumscribed to the circle at infinity. But he introduced the imaginaries only by their symmetric functions and in consequence was not able to define the anharmonic ratio of four elements when these ceased to be real in whole or in part. If Chasles had been able to establish the notion of the anharmonic ratio for imaginary elements, a formula which he gives in the Géométrie supérieure (page 118 of the new edition) would have given him immediately that beautiful definition of an angle as the logarithm of an anharmonic ratio, which enabled Laguerre, our regretted colleague, to give the complete solution so long sought to the problem of the transformation by homography and correlation of the relations which contain both angles and segments.

As did Chasles, so also has Steiner, that great and profound mathematician, followed the path of pure geometry; but he has neglected to give us a complete exposition of the methods upon which he relied. However, they can be characterized by saying that they rested upon the introduction of those elementary geometric forms which Desargues had already considered; upon the development which he was capable of giving to Bobillier's theory of polars; and finally upon the construction of higher curves and surfaces by the aid of pencils or nets of lower orders. Even if this were not known from more recent researches, analysis would suffice to show us that the field thus covered is coextensive with that to which the method of Descartes gives easy access.

IV.

While Chasles, Steiner, and later, as we shall see, von Staudt, applied themselves to the task of constructing a rival doctrine to analysis, thus in a way setting up one altar against another, Gergonne, Bobillier, Sturm, and above all Plücker, were perfecting the cartesian geometry and developing an analytic system somewhat adequate to the discoveries of the geometricians. It is to Bobillier and to Plücker that we owe the so-called method of abridged notation. Bobillier devoted to this method some truly original pages in the later volumes of Gergonne's Annales. Plücker began to develop it in his first volume, soon followed by a series of works in which are established the foundations of methodically modern analytic geometry. To him we owe tangential coördinates, trilinear coördinates, used in homogeneous equations, and finally the employment of canonical forms, the validity of which is recognized by the so-called method of the enumeration of constants, often misleading but very productive. All these happy inspirations served to infuse new blood into cartesian analysis and to enable it to bring out the full meaning of those conceptions which the so-called synthetic geometry had not been able to master completely. Plücker, with whom in this connection it is doubtless just to join Bobillier (who died prematurely) should be considered as the true originator of those methods in modern analysis in which the use of homogeneous coördinates permits one to treat simultaneously and, so to speak, without the reader noticing it, not only the particular figure under consideration, but also all those deduced from it by homography and correlation.

V.

From this time on a brilliant period opened for geometric research of every kind. The analysts interpreted all their results and endeavored to translate these results in their constructions. The geometricians aimed at discovering in every problem some general principle, usually demonstrable only with the aid of analysis, so that from this principle might be deduced without effort a mass of particular consequences, solidly bound together, and bound also to the principle from which they were derived. Thus Otto Hesse, the brilliant disciple of Jacobi, developed in an admirable manner that method of homogeneous variables to which Plücker perhaps had not been able to give full value. At this time also Boole discovered in Bobillier's polars the first notion of a covariant; the theory of quantics was created by the works of Cayley, Sylvester, Hermite, and Brioschi. Later Aronhold, Clebsch and Gordan, and other geometers still living, furnished the final notations, established the fundamental theorem relative to the limitation of the number of independent covariant forms, and thus gave that theory its present completeness.

The theory of surfaces of the second order, built up mainly by the school of Monge, was enriched by a mass of elegant properties, established for the most part by Otto Hesse, who found later in Paul Serret a worthy rival and successor.

The properties of the polars of algebraic curves were developed by Plücker, and still more by Steiner. The study of the cubic, which dates back to a much earlier period, was rejuvenated and enriched by a mass of new material. Steiner was the first to study by the methods of pure geometry the double tangents of the quartic, and later Hesse applied the methods of algebra to this beautiful question as well as to that of the points of inflexion of the cubic.

The notion of the class of a curve, introduced by Gergonne, and the study of a paradox partly elucidated by Poncelet, relative to the respective degrees of two curves, polar reciprocals to each other, gave birth to the researches of Plücker relative to the so-called ordinary singularities of algebraic plane curves. The celebrated formulas to which Plücker was thus led were later extended by Cayley and other geometricians to algebraic curves in space, and by Cayley and Salmon to algebraic surfaces. The singularities of higher orders were in their turn taken up by the geometers; contrary to an opinion then quite prevalent, Halphen demonstrated that not every such singularity can be considered as equivalent to a definite aggregate of ordinary singularities. These researches closed for a time this difficult and important question.

Analysis and geometry — Steiner, Cayley, Salmon, and Cremona, met in the study of cubic surfaces, and conformably to the anticipation of Steiner this theory became as simple and easy as that of the surfaces of the second order.

The algebraic ruled surfaces, so important in the applications, were studied by Chasles, by Cayley whose influence and footsteps are found in all mathematical investigations, by Cremona, by Salmon, by La Gournerie, and later by Plücker in a work to which we shall return later.

The study of the general surface of the fourth order seemed still to be too difficult; but that of certain particular surfaces of this order with multiple points or lines was commenced by Plücker for the wave surface, by Steiner, Kummer, Cayley, Moutard, Laguerre, Cremona, and many other investigators. As to the theory of algebraic curves in space, enriched in the elementary parts, it received the most marked growth later through the labors of Halphen and Noether, whom it is impossible here to separate. A new theory destined to have a great future was born with the works of Chasles, Clebsch, and Cremona; it concerns the study of all algebraic curves which can be traced upon a given surface.

Homography and correlation, the two methods of transformation which were the distant origin of all preceding researches, received from them in turn unlooked-for assistance and augmentation: They are not the only transformations that make a single element correspond to a single element, as could have been shown by a particular transformation cursorily treated by Poncelet in the Traité des propriétés projectives. Plücker defined the transformation by reciprocal radii vectores or inversion, whose great importance Sir William Thomson and Liouville were not long in showing both in mathematical physics and in geometry. A contemporary of Möbius and of Plücker, Magnus, thought he had found the most general transformation which makes one point correspond to one point, but the investigations of Cremona show us that Magnus's transformation is only the first term of a series of birational transformations which the great Italian geometer shows us how to determine methodically, at least for the figures of plane geometry. Cremona's transformations will long preserve their importance and interest even though later investigations have shown that they always reduce to a series of successive applications of Magnus's transformation.

VI.

Every one of the works just enumerated, together with others to which we shall return later, finds its origin and, at least to a certain degree, its impetus in the conceptions of modern geometry; but it is now fitting to indicate rapidly another source of great progress in the study of geometry. Legendre's theory of the elliptic functions, sadly neglected by French geometricians, was developed and enlarged by Abel and Jacobi. In the hands of these great mathematicians, soon followed by Riemann and Weierstrass, the theory of abelian functions, which algebra later endeavored to handle alone, began to bring to the geometry of curves and surfaces a contribution the importance of which will continue to grow.

Even earlier, Jacobi had used the theory of elliptic functions in the demonstration of Poncelet's celebrated theorems relating to inscribed and circumscribed polygons, introducing thus a chapter which has since been enriched by a multitude of elegant results; already also had he obtained, by methods belonging to geometry, the integration of the abelian equations.

But it was left to Clebsch to be the first to show in a long series of articles the whole importance of Abel's and Riemann's notion of the deficiency of a curve by developing numerous results and elegant solutions in which the use of abelian integrals produced such simplicity as to make this appear their true origin. The study of the inflexional points of the cubic curve, that of the double tangents to the quartic, and in general that of the theory of osculation which has furnished interest to so many both in earlier and recent times, was linked to the beautiful problem of the division of the periods of elliptic and abelian functions.

In one of his memoirs, Clebsch had studied the rational curves or those of deficiency zero; this led him toward the end of his regretably short life to consider what one may call rational surfaces, namely those which may be simply represented upon a plane. Here there was a great field for discovery, opened already for the elementary cases by Chasles, in which Clebsch was followed by Cremona and many other scholars. This was the occasion when Cremona, generalizing his investigations in plane geometry, brought to light if not the totality of birational transformations of space, at least some of the more interesting among these transformations. The extension of the idea of deficiency to algebraic surfaces was already begun; and some later works of great value have shown that the theory of simple or multiple integrals of algebraic differentials, in the consideration of surfaces as well as curves, will offer an extended region of important applications; but it is not in a historical account of geometry that it is proper to emphasize this matter.

VII.

While thus those mixed methods whose principal applications we have just indicated were being founded, those interested in pure geometry were not inactive. Poinsot, the creator of the theory of couples, developed by a purely geometric method, "in which," as he said, "not for a single moment is the object of the investigation lost sight of," the theory of the rotation of a solid body which the researches of d'Alembert, Euler, and Lagrange seemed to have exhausted; Chasles made a valuable contribution to kinematics in his beautiful theorems on the displacement of a solid body, which have been extended since by other elegant methods to the case where the movement has various degrees of freedom. He made known also his beautiful propositions on attraction in general, which stand without disadvantage beside those of Green and Gauss. Chasles and Steiner met in the study of the attraction of the ellipsoid and thus showed once more that geometry holds a conspicuous place in the highest questions of the integral calculus.

Steiner did not disdain to occupy himself at the same time with the elementary parts of geometry. His investigations on the contact of circles and conics, on isoperimetric problems, on parallel surfaces, and on the centroid of curvature excited the admiration of everyone by their simplicity and depth.

Chasles introduced his principle of correspondence between variable elements of two systems which has given birth to so many applications; but here again analysis claims its place to study the principle in its essence, to give it its exact form, and finally to generalize it. Exactly similar is the history of the famous theory of characteristics, and of the numerous researches of de Jonquières, Chasles, Cremona, and others, which were to furnish the foundation of a new branch of the science, viz., enumerative geometry. For many years the celebrated postulate of Chasles was admitted without objection; many geometers thought they had established it in an irrefutable manner. But, as Zeuthen then said, it is extremely difficult to make sure in demonstrations of this kind that there is not still some weak place in the reasoning which the author has not perceived; and indeed, Halphen, after some fruitless attempts, finally capped all these researches by showing clearly in what cases Chasles's postulate can be admitted, and when it must be rejected.

VIII.

Such were the principal investigations which at that time reinstated synthetic geometry in its honorable position, and assured to it during the last century the place which belongs to it in mathematical research. Numerous and illustrious laborers took part in this great geometric movement, but it must be acknowledged that it had Chasles and Steiner as its leaders. Such was the brilliancy displayed by their marvelous discoveries that, at least for a time, they threw into the shade the publications of many modest geometricians, less preoccupied perhaps in finding brilliant applications which should make geometry attractive, than in fixing this science upon an absolutely solid foundation. The work of these latter has perhaps received more tardy recognition, but their influence increases each day; and it will doubtless increase still more. To pass them over in silence would be to neglect, doubtless, one of the principal factors which will play a part in future investigations. It is primarily to von Staudt that we allude here. His geometric work was presented in two publications of great merit: the Geometrie der Lage, appearing in 1847, and the Beiträge zur Geometrie der Lage, published in 1856, nearly four years after the Géométrie supérieure.

Chasles, as we have seen, was preoccupied with founding a body of doctrine independent of the cartesian analysis, and here he was not entirely successful. We have already indicated one of the objections that can be raised against this system, viz: imaginary elements are there defined only by their symmetric functions, and this excludes them necessarily from very many researches. On the other hand, the constant use of the anharmonic ratio, of transversals, and of involution, which requires frequent analytic transformations, gives to the Géométrie supérieure an almost exclusively metrical character which separates it notably from the methods of Poncelet. Returning to those methods, von Staudt applied himself to founding a geometry free from every metric relation and based exclusively on relations of position. It was in this spirit that his first work, the Geometrie der Lage of 1847, was conceived. The author there takes as starting point the harmonic properties of the complete quadrilateral and those of homologous triangles, proved solely by theorems of three-dimensional geometry wholly analogous to those of which Monge's school made such frequent use.

In this first part of his work von Staudt entirely neglected imaginary elements. Only in the Beiträge, his second work, by means of a very original extension of Chasles's method, did he succeed in defining geometrically an isolated imaginary element and in distinguishing it from its conjugate. This extension, while rigorous, is laborious and very abstract. In substance it can be explained as follows: Two conjugate imaginary points can always be considered as the double points of an involution on a real straight line; and just as we pass from one imaginary to its conjugate by changing i into $-i$, so we may distinguish the two imaginary points by making correspond to each of them one of the two different senses of direction that can be attributed to the line. There is something a little artificial in this; and the development of the theory raised on such a foundation is necessarily complicated. By purely projective methods von Staudt established a complete method of reckoning with the anharmonic ratio of the most general imaginary elements. Projective geometry, like all the rest of geometry, uses the idea of order, and order begets number; it is not then astonishing that von Staudt was able to create this calculus of his; and we must admire the ingenuity which he displayed in attaining this end. Despite the efforts of the distinguished geometers who have endeavored to simplify his presentation, we fear that this part of von Staudt's geometry, as well as the very interesting geometry of the profound thinker Grassmann, cannot prevail against the analytic methods which now have acquired almost universal favor. Life is short, the geometrician knows and practices the principle of least action. Despite these fears, which should not discourage any one, it seems to us that under the first form in which it was presented by von Staudt projective geometry should become the necessary companion to descriptive geometry, that it is destined to renew this geometry in its spirit, in its methods of procedure, and in its applications. This has been already recognized in many countries, notably in Italy, where the great geometer Cremona did not disdain to write an elementary treatise on projective geometry for the schools.

IX.

In the preceding sections we have endeavored to follow through and to point out clearly the far-reaching consequences of the methods of Monge and Poncelet. In originating tangential and homogeneous coördinates Plücker seemed to have exhausted all that the methods of projections and reciprocal polars could furnish to analysis. It remained for him, toward the end of his life, to return to his earlier investigations and give to them an extension which should enlarge in unlooked-for proportions the domain of geometry.

Preceded by numerous researches upon systems of straight lines, due to Poinsot, Möbius, Chasles, Dupin, Malus, Hamilton, Kummer, Transon, and above all to Cayley (who first introduced the idea of the coördinates of a line), investigations which have their origin partly in statics and kinematics, partly in geometrical optics — though preceded by all these, Plücker's geometry of the straight line will always be regarded as the part of his work where the newest and most interesting ideas are found. The fact that Plücker was the first to create a methodic study of the straight line is in itself important, but is nothing compared to what he discovered. It is sometimes said that the principle of duality puts in evidence the fact that the plane can be considered as an element of space just as well as the point. This is so, but in adding to the plane and to the point the straight line as a possible element of space Plücker was led to realize that any curve whatever, that any surface whatever can also be considered as an element of space, and thus is brought into existence that new geometry which already has inspired a great number of investigations and which will stimulate the authors of still more in the future. A beautiful discovery of which we shall speak later has already connected the geometry of spheres with that of straight lines, and has led to the introduction of the idea of the coördinates of a sphere. The theory of systems of circles is already begun; it will doubtless be developed further when it is found desirable to study the representation, due to Laguerre, of an imaginary point in space by means of an oriented circle.

But before presenting the development of these new ideas which have vitalized Monge's infinitesimal methods, it is necessary to turn back and pick up again the history of the branches of geometry so far neglected in this discussion.

X.

Thus far we have confined ourselves to the consideration of those investigations of the school of Monge which have to do with finite geometry; but some of Monge's disciples devoted themselves primarily to the development of those new ideas in infinitesimal geometry that had been applied by their master to curves of double curvature, to lines of curvature, and to the generation of surfaces, ideas which are developed at least in part in the Application de l'Analyse à la Géométrie. Among these men we must cite Lancret, the author of the beautiful researches on twisted curves, and above all Charles Dupin, perhaps the only one who followed in all the paths opened by Monge.

Besides other publications, we owe to Dupin two works, of which Monge himself would not have been ashamed to be the author; the Développements de Géométrie pure, appearing in 1813, and the Applications de Géométrie et de Mécanique, which dates from 1822. Here is found that idea of the indicatrix which, after Euler and Meunier, was to revive and refound completely the theories of curvature, of conjugate tangents, and of asymptotic lines, which have taken such an important place in recent investigations. Nor should we forget the determination of the surface upon which all lines of curvature are circles, nor above all the memoir on triply orthogonal systems of surfaces, where may be found not only the discovery of the triple system of surfaces of the second degree but also the celebrated theorem to which the name of Dupin will always be attached.

Under the influence of these publications and of the revival of synthetic methods, the geometry of the infinitely small again assumed in all investigations the place from which Lagrange had wished to exclude it forever. It is singular that the geometric methods thus restored were about to receive the liveliest impulse from the publication of a memoir which, at least at first sight, seemed to belong to the purest analysis; we allude to Gauss's celebrated paper, "Disquisitiones generales circa superficies curvas," which was presented in 1827 to the Royal Society at Göttingen, and whose appearance can well be said to mark an important and eventful date in the history of the infinitesimal geometry.

Thanks to this powerful help and to the writings of Monge and Dupin, the infinitesimal method received in France an impetus before unknown. Frenet, Bertrand, Molins, J. A. Serret, Bouquet, Puiseux, Ossian Bonnet, and Paul Serret developed the theory of twisted curves. Liouville, Chasles, and Minding joined them in the prosecution of the methodic study of Gauss's memoir. Jacobi's integration of the differential equation of the geodesic lines of the ellipsoid also incited many investigations. At the same time there was a great development of the problems studied in Monge's Application de l'Analyse. Some of the results already partially obtained by Monge were completed in the happiest way by the determination of all the surfaces having plane or spherical lines of curvature.

At this time lived Gabriel Lamé, one of the most acute of geometers, according to the judgment of Jacobi. He, like Charles Sturm, had commenced with pure geometry and had made most interesting contributions to this science by means of a little book published in 1817 and by his memoirs in Gergonne's Annales. Employing the results of Dupin and Binet concerning the system of confocal surfaces of the second degree, and rising to the notion of curvilinear coördinates in space, he became the inventor of a whole new theory destined to receive the most varied applications in the domain of mathematical physics.

XI.

Here also, in infinitesimal geometry, are found the two tendencies which have been noticed in the geometry of finite quantities.

Some, among whom must be placed J. Bertrand and O. Bonnet, wished to build up an independent method resting directly upon the use of the infinitely small. Bertrand's great Traité de Calcul différentiel contains several chapters on the theory of curves and surfaces which to a certain extent illustrate this point of view. The exponents of the other tendency followed the ordinary analytic way, aiming only at thoroughly understanding and exhibiting the elements which are of primary importance. This is the course which Lamé followed in introducing his theory of differential parameters. This is the course which Beltrami followed in extending with great ingenuity the use of these differential invariants to the case of two independent variables, that is to say, to the study of surfaces.

At present it seems to be the custom to prefer a mixed method, the origin of which is found in Ribaucour's writings under the name of *perimorphy*. Here the rectangular axes of analytic geometry are retained, but they are rendered movable and related with the system to be studied in the manner which appears the most advantageous. Most of the objections urged against the method of coördinates are thus removed; and the advantages of what is sometimes called intrinsic geometry are combined with those which result from the use of the ordinary analysis. Nor is this analysis abandoned; but the complicated calculus which it almost always introduces when applied to the study of surfaces in rectilinear coördinates disappear soonest if we use the notions of invariants and covariants of differential quadratic forms, which we owe to the investigations of Lipschitz and Christoffel that were inspired by Riemann's study of non-euclidean geometry.

XII.

The results of these numerous investigations were not long in coming. The notion of geodesic curvature, which Gauss already had found but had not published, was given by Bonnet and Liouville; the theory of surfaces with a functional relation between the radii of curvature, begun in Germany with two propositions which would have worthily occupied a place in Gauss's memoir, was enriched with a mass of propositions by Ribaucour, Halphen, S. Lie, and others. Some of these propositions have to do with the general properties of such surfaces; others have to do with particular cases where the relation between the radii of curvature takes a particularly simple form, *e. g.*, the minimal surfaces, and surfaces with constant curvature, positive or negative.

Minimal surfaces have been the object of investigations which have made their study the most attractive chapter in infinitesimal geometry. The integration of their partial differential equation forms one of the most beautiful discoveries of Monge; but, owing to the imperfection of the theory of the imaginary, that great geometrician could not get from his formulas any method of generation of these surfaces — nor even a single particular surface. We will not here enter into the detailed history which has been presented in our Leçons sur la théorie des surfaces; but it is fitting to recall the fundamental investigations of Bonnet which have given to us, in particular, the idea of the surfaces *associated* with a given surface; also Weierstrass's formulas which establish a close bond between minimal surfaces and the functions of a complex variable; also Lie's investigations by which it has been established that even Monge's formulas could now be made the basis of a profitable study of minimal surfaces. The endeavor to determine the minimal surfaces of the lowest class or degree led to the notion of double minimal surfaces, which belongs to analysis situs.

Three problems varying in importance have been studied in this theory.

The first, relative to the determination of the minimal surfaces inscribed to a given developable with a given curve of contact, was solved by those celebrated formulas which later led to many special propositions; *e. g.*, every straight line traced upon such a surface is an axis of symmetry.

The second, formulated by S. Lie, requires the determination of all algebraic minimal surfaces inscribed in an algebraic developable, the curve of contact not being given. This problem also has been completely solved.

The third and most difficult is that problem which the physicist solves empirically by plunging a closed contour into a solution of glycerine. The problem has to do with the determination of the minimal surface passing through a given contour. It is evident that the solution surpasses the resources of geometry. But thanks to the capabilities of the highest analysis, it has been solved for certain contours in the celebrated memoir of Riemann and in the profound researches which followed or were coeval with this memoir. As to the solution for a perfectly general contour, its study has been brilliantly begun and it will be continued by our successors.

After minimal surfaces, the surfaces with constant curvature naturally attracted the attention of geometers. Bonnet's ingenious theorem connected together the surface of constant mean curvature and the surface of constant total curvature. Bour announced that the partial differential equation of the surface of constant curvature could be completely integrated. This result has not been attained; and its attainment seems hardly probable in view of the investigation in which Sophus Lie vainly tried to apply a certain general method of integrating partial differential equations to the particular equation of surfaces with constant curvature. But while it is impossible to determine all these surfaces in finite terms, at least some of them characterized by special properties have been obtained, namely those with plane or spherical lines of curvature; and by the use of a method which has proved successful in many other problems, it has been shown that from every surface with constant curvature can be derived an infinity of other surfaces of the same kind, and this by the use of clearly defined operations which require only quadratures.

The theory of the deformation of surfaces in Gauss's sense was also greatly enriched. To Minding and Bour is due the detailed study of that special deformation of ruled surfaces which leaves the generators straight lines. While, as has just been said, the determination of surfaces applicable upon a sphere was impossible, this determination for other surfaces of the second degree has been attacked with greater success, and particularly for the paraboloid of revolution. The systematic study of the deformation of the general surfaces of the second degree is already opened; it is a study which will doubtless give most important results in the future.

The theory of an infinitely small deformation constitutes today one of the most complete chapters of geometry. It is the first extended application of a general method which seems to have a great future.

Being given a system of differential equations or a system of partial differential equations, sufficient to determine a certain number of unknowns, we conventionally associate with that system another system of equations, which we have called the auxiliary system, and which determines the systems of solutions infinitely near to any given system of solutions. The auxiliary system being necessarily linear, its use in any investigation throws a valuable light on the properties of the proposed system and upon the possibility of its integration.

The theory of lines of curvature and of asymptotic lines has been notably extended. Not only was it possible to determine these two sets of lines for certain particular surfaces such as Lamé's tetrahedral surfaces; but also, in developing Moutard's results relative to a particular class of linear partial differential equations of the second order, it was possible to generalize what had been obtained for surfaces with plane or spherical lines of curvature by determining completely all the classes of surfaces for which the problem of the spherical representation can be solved. The correlated problem relative to the asymptotic lines was solved also by finding all the surfaces of which the infinitely small deformation can be determined in finite terms. There is here a vast field for investigation, the exploration of which has hardly been begun.

The infinitesimal study of congruences of straight lines, begun long since by Dupin, Bertrand, Hamilton, and Kummer, is only beginning to be connected with all these investigations. Ribaucour, who in this took a leading part, studied certain classes of rectilinear congruences and in particular the so-called isotropic congruences, which come in most happily in the study of minimal surfaces.

The triply orthogonal systems which Lamé had employed in mathematical physics became the object of systematic investigations. Cayley was the first to form the partial differential equation of the third order upon which the general solution of this problem has been made to depend. The system of confocal surfaces of the second order was generalized and gave birth to the theory of the general cyclides, in which theory it is possible to use at one and the same time the resources of the metrical, projective, and infinitesimal geometries. Many other orthogonal systems were presented. Among these it is fitting to mention Ribaucour's cyclic systems for which one of the three families has circles for its orthogonal trajectories, and the more general systems for which these orthogonal trajectories are simply plane curves. The systematic use of the imaginary, which must not be excluded from geometry, rendered it possible to make all these determinations depend upon the study of the finite deformation of a particular surface.

Among the methods which made possible all these results, it is fitting to designate the systematic use of the linear partial differential equations of the second order and of systems of such equations. The most recent investigations show that this method is destined to lead to a revision of most of the theories.

Infinitesimal geometry could not neglect the study of the two fundamental problems set by the calculus of variations.

The problem of the shortest path on a surface was the object of masterly studies by Jacobi and Ossian Bonnet. The study of the geodesic lines was extended, and these lines were determined for new surfaces. The theory of assemblages made it possible to follow these lines in their course on a given surface. The solution of a problem relative to the representation of one surface upon another increased greatly the interest in the discoveries of Jacobi and Liouville relative to a particular class of surfaces where the geodesic lines can be determined. The results from this particular case lead to the consideration of a new question, namely : To find all the problems of the calculus of variation the solutions of which are furnished by curves satisfying a given differential equation.

Finally, Jacobi's methods were extended to space of three dimensions and applied to obtain the solution of a question which presents the greatest difficulties, namely : the study of the minimum properties belonging to a minimal surface passing through a given contour.

XIII.

Among the discoverers who have contributed to the development of infinitesimal geometry, Sophus Lie is distinguished by many important discoveries which place him in the first rank. He was not one of those who exhibit in youth very marked aptitudes ; when in 1865 he left the University of Christiania, he was still hesitating between philology and mathematics. The works of Plücker first caused him to realize his true vocation ; and he published in 1869 a first investigation on the interpretation of imaginaries in geometry, and from 1870 on he was in possession of the ideas which mapped out his whole career.

I had the pleasure at that time of frequently seeing and entertaining him in Paris, whither he had come with his friend Felix Klein. A course of lectures delivered by Sylow and attended by Lie disclosed to him the great importance of the theory of substitutions ; and the two friends studied this theory in Camille Jordan's great treatise; they recognized fully the important rôle which this theory was destined to play in many branches of mathematical science where it had not yet been applied. Each of them has had the good fortune of contributing by his publications to turning mathematical research in that direction which seemed to him the best.

As early as 1870 Sophus Lie presented to the Paris Academy of Sciences an extremely interesting discovery: Nothing resembles a sphere so little as a straight line, but nevertheless Lie conceived of a peculiar transformation which made a sphere correspond to a straight line, and therefore set up a method of linking every proposition having to do with straight lines to a proposition having to do with spheres and *vice versa*. In this most striking method of transformation each property of the lines of curvature of a surface furnishes a proposition on the asymptotic lines of the transformed surface. Lie's name will always remain attached to those recondite relations which join together those two essential and fundamental elements of geometrical investigation, the straight line and the sphere. These relations he developed in a memoir filled with new ideas, which appeared in 1872.

The researches which followed Lie's brilliant entrance into mathematics fully confirmed the hopes to which it had given rise. Plücker's conception of the generation of space by straight lines, or by arbitrarily chosen curves or surfaces, opened to the theory of algebraic forms a field which was not yet explored, which Clebsch had barely started to investigate and for which he had only just begun to fix the boundaries. But in the domain of the infinitesimal geometry this conception was given its full value by Sophus Lie. The great Norwegian geometrician then found first the notion of congruences and complexes of curves and later that of contact transformations, the first germ of which he had found for the case of the plane in Plücker. The study of these transformations led him to perfect, contemporaneously with Mayer, the integration methods which Jacobi had instituted for partial differential equations of the first order; moreover it threw the most radiant light upon the most difficult and obscure parts of the theory of partial differential equations of a higher order. In particular it allowed Lie to indicate every case in which Monge's method of characteristics is fully applicable to equations of the second order in two independent variables.

In the continuation of the study of these special transformations, Lie was led to construct step by step his masterly theory of continuous transformation groups and to render evident the important role which the group notion plays in geometry. Among the essential elements of his investigations it is fitting to mention the infinitesimal transformations, the idea of which belongs exclusively to him. Three great works published under his direction by able and devoted collaborators contain the essential part of his labors together with their application to the theory of integration, to the theory of complex units, and to non-euclidean geometry.

XIV.

I have thus arrived by an indirect way at the non-euclidean geometry, the study of which daily assumes increasing importance in the researches of geometricians. If I were the only speaker on geometry, I should take pleasure in recalling to you everything that has been done in this subject since Euclid or at least since Legendre, up to the present day. Considered in succession by the greatest geometricians of the last century, this subject has been progressively enlarged. It began with the consideration of the celebrated parallel postulate, but it has come to include all the geometric axioms.

Euclid's Elements, which have withstood the toil of so many centuries, will at least have had the honor of calling forth, before their end, a long series of admirably connected investigations, which not only will contribute in a most efficacious way to mathematical progress, but also will furnish to the philosopher the most precise and firm starting points for the study of the origin and formation of our concepts. I am assured in advance that my distinguished collaborator will not forget among the problems of the present this problem, which perhaps is the most important, and with which he has occupied himself with so much success; and I leave to him the task of developing it with all the fullness which it assuredly deserves.

I have just spoken of the elements of geometry. They have received during the last hundred years additions which it is not right to forget. The theory of polyhedrons was enriched by Poinsot's beautiful discoveries on star-shaped, and by Möbius's one-sided polyhedrons. The transformation methods have enlarged the aspect of this study. We can to-day express our view of this subject by saying: That the first book contains the theory of translation and symmetry, that the second is equivalent to the theory of rotation and translation, that the third depends upon inversion and the theory of figures similar and similarly placed. But it must be acknowledged that it is by the aid of analysis that the elements have been enriched by their most beautiful propositions. For it is to the highest analysis that we owe the solution of the problem of the inscribed regular polygon of 17 sides and of the analogous problems. To it we owe the demonstrations so long sought of the impossibility of the quadrature of the circle, of the impossibility of certain geometric constructions by the aid of the straight-edge and compasses only. Finally, it is to it that we owe the first rigorous demonstrations of the maximum and minimum properties of the sphere. It will be the task of geometry to enter this domain where analysis has preceded it.

What will be the elements of geometry in the course of the century just beginning? Will there be a single elementary book of geometry? America perhaps, with its schools freed from every program and tradition, will give us the best solutions of this important and difficult question. Von Staudt has sometimes been called "the Euclid of the nineteenth century"; I should prefer to call him "the Euclid of projective geometry"; but is this branch of geometry, no matter how interesting it may be, called on to furnish alone the foundation of the future Elements?

XV.

The time has come to close this narrative already too long, and yet there remains a mass of interesting investigations which I may be said to have neglected under compulsion. I should have liked to speak to you of those geometries in any number of dimensions, the idea of which goes back to the beginnings of algebra, but of which the systematic study has been taken up only in the last sixty years by Cayley and Cauchy.

This kind of research has found favor in your country, and I need hardly recall to you that our president,* after showing himself the worthy continuer of Laplace and Leverrier, in a space which he considers with us as being endowed with three dimensions, has not disdained to publish in the *American Journal* considerations of great interest on the geometries of n dimensions. A single objection already formulated by Poisson may

* Professor Simon Newcomb.

be urged against studies of this kind: namely, the absence of all real basis, of any substratum that will allow the results obtained to be presented under a visible and in a way tangible form. The extension of the methods of descriptive geometry, and above all the employment of Plücker's notions of the generation of space will do much toward depriving this objection of its weight.

I should also have wished to speak to you of the method of equipollences, the germ of which we find in Gauss's posthumous papers, of Hamilton's quaternions, of Grassmann's methods, and in general of the systems of complex units, of the analysis situs so closely connected with the theory of functions, of the so-called kinematic geometry, of the theory of abaci, of geometrography, of geometric applications in natural philosophy and in the arts. But I would fear if I enlarged too much, that some analyst — there have been such — would accuse geometry of wishing to monopolize everything.

My admiration for analysis, become so fruitful and powerful in our time, would not permit me to conceive such a thought. But if such a reproach could be formulated to-day, I think that it should be raised, not against geometry but against analysis. The circle in which mathematical studies seemed bound at the beginning of the nineteenth century has been broken on all sides. The old problems present themselves to us under an altered form. New problems arise studied by legions of workers. The number of those who cultivate pure geometry has become very limited. This is a danger against which it is of some importance to guard. Do not let us forget that while analysis has acquired means of investigation which formerly it lacked, nevertheless it owes those means largely to the concepts introduced by the geometricians. Geometry must not remain, as it were, shrouded in its own triumph. It was in the school of geometry that we have learned, and there our successors will have to learn it, never blindly to trust to too general methods, but to consider each question on its own merits, to find in the particular conditions of each problem either a direct way toward a simple solution or the means to apply in an appropriate manner those general methods which every science should collect. Thus, as Chasles says in the beginning of the Aperçu historique: "The doctrines of pure geometry often, and in many questions, give a simple and natural way to penetrate to the origin of truths, to lay bare the mysterious chain which unites them, and to make them known individually, luminously and completely." Therefore let us cultivate geometry which has its own advantages, and this without wishing to make it equal in all points to its rival. Besides, if we should be tempted to neglect it, it would soon find, as it once has done, in the applications of mathematics the means of reviving and developing itself anew. It is like the giant Antæus, who regained his strength whenever he touched his mother earth.

NOTE ON FERMAT'S NUMBERS.

BY DR. J. C. MOREHEAD.

(Read before the American Mathematical Society, April 29, 1905.)

How many primes are contained in the form

$$F_n = 2^{2^n} + 1$$

is a famous question in the theory of numbers. Fermat showed that F_0, F_1, $\cdots$, F_4 are primes, and stated that he believed F_n to be prime for every value of n. This statement of Fermat was generally accepted as correct until Euler showed in 1732 that F_5 has the factor 641. In the period 1878–1903 factors were found of eight other Fermat numbers, viz.,

$$F_6, \quad F_9, \quad F_{11}, \quad F_{12}, \quad F_{18}, \quad F_{23}, \quad F_{36}, \quad F_{38}.$$

In the present note we add to the list the tenth composite case, F_7, identified by applying Pepin's criterion:* "The necessary and sufficient condition that F_n be prime is

$$P_n = \alpha^{\frac{1}{2}(F_n - 1)} + 1 \equiv 0 \pmod{F_n},$$

α being any quadratic non-residue of F_n." Taking the non-residue $\alpha = 3$, I have found by a lengthy computation just completed that

* *Comptes rendus*, vol. 85 (1878), p. 329.