we obtain surfaces of translation, applicable to one another and depending upon seven arbitrary parameters.

The conditions that there exist three constants a, b, c such that for all values of V and V_1 the curves $u = $ const. lie in the planes $aX + bY + cZ = d$ are

$$ak - b - cg = 0,$$

$$a(hb_3 - kb_2) + b(kb_1 - gb_3) + c(gb_2 - hb_1) = 0,$$

$$a[h + (hc_3 - kc_2)e^{-a}] + b[-g + e^{-a}(kc_1 - gc_3)]$$
$$+ c[1 + e^{-a}(gc_2 - hc_1)] = 0.$$

Equating to zero the determinant of these equations, we get in consequence of (19) the following equations of conditions

$$(22) \quad (k^2 + h^2)b_1 - (k + gh)b_2 + (h - kg)b_3 =$$
$$h(ga_1 + ha_2 + ka_3)e^{-a},$$

and the planes of the curve are parallel to the plane

$$[(h + gk)b_1 - gb_2 - g^2b_3]x + h(kb_1 - gb_3)Y +$$
$$[k^2b_1 - kb_2 + (h - kg)b_3]Z = 0.$$

As in the particular case previously considered, equation (22) is the condition also that the curves $v = $ const. on S_1' lie in parallel planes.

Another seven parameter aggregate of pairs of applicable surfaces of translation is found when the values from (1) and (2) are substituted in equations (20) and (21).

PRINCETON UNIVERSITY,
      February, 1905.

THE GROUPS OF ORDER 2^m WHICH CONTAIN AN INVARIANT CYCLIC SUBGROUP OF ORDER 2^{m-2}.

BY PROFESSOR G. A. MILLER.

HALLET [*] has recently called attention to the fact that Burnside omits one group in his enumeration of the non-abelian groups of order 2^m which contain an invariant cyclic subgroup

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[*] BULLETIN, vol. 11 (1905), p. 318; *Science*, vol. 21 (1905), p. 176.

A of order 2^{m-2} such that the corresponding quotient group is cyclic. There are six such groups which do not include an operator of order 2^{m-1} while Burnside gives only five of them. *   There are however only eight possible non-abelian groups of order 2^m which do not include an operator of order 2^{m-1} and give rise to a non-cyclic quotient group with respect to A, while Burnside's statement that the total number of groups in question is fourteen implies that there are nine such groups.

It seems desirable to give the characteristic properties of all the groups in question in order that the reader of Burnside may be able to verify the double error with ease and also to complete the list of the groups of order 2^m which include operators of order 2^{m-2}. The ten groups given in § 1 are included in the more extensive enumeration of all the groups of order 2^m which include the abelian group of type $(m-2, 1)$. †   In particular, the group given by Hallet is G_{16} of this list. It is, however, somewhat difficult in several instances, to determine which of the groups of this list contain an invariant cyclic subgroup of order 2^{m-2} since the groups were determined from a different standpoint. On this account it seems desirable to include these groups in the present enumeration.

For detailed explanation of the method employed to prove that no other groups exist we would refer to the list mentioned in the preceding paragraph and also to a brief article in the *American Journal*, volume 24 (1902), page 395, " On a method of constructing all the groups of order p^m." In what follows it will be assumed that the groups in question are non-abelian and of order 2^m, include A, and do not include any operator of order 2^{m-1}.

Each of the groups under consideration contains at least one subgroup of order 2^{m-1} including A. Such a subgroup will be denoted by H while G will be used to represent the entire group. The symbol $G-H$ will be employed to represent the operators of G which are not also in H. The groups will be considered under two headings, according as H is abelian or non-abelian. Those under each heading will be subdivided into two categories, according as G/A is non-cyclic or cyclic.

§ 1. *Groups in which H is abelian.*

The type of H is $(m-2, 1)$ and the subgroup I_1 of its group of isomorphisms, which is composed of all the operators which transform A into itself, is the direct product of the group of isomorphisms of A and an operator of order two. Hence I_1 is abelian and of type $(m-4, 1, 1)$. The operators in $G-H$ must transform H according to an operator of order 2 contained in I_1. If t represents such an operator G/A is cyclic or non-cyclic as t^2 is in $H-A$ or in A. We begin with the latter case.

As H contains two cyclic groups of order 2^{m-2} which are invariant under G, it is necessary to consider only one from each of the two pairs of operators of order 2 in I_1 which are conjugate under the group of isomorphisms of H.* That is, we have to consider only five of the seven operators of order two contained in I_1, viz., the three which transform each operator of H into the same power, the one which is commutative with every operator of A, and the one which transforms each operator of A into its inverse while it transforms the operators of $H-A$ into their inverses multiplied by the operator of order 2 contained in A. Each of the last two has two conjugates under the group of isomorphisms of H.

The two groups (G_1, G_2) which transform each operator of H into its inverse contain respectively 2^{m-1} operators of order 2, or 2^{m-1} operators of order 4 in addition to H. The group G_3 which transforms each operator of H into its $2^{m-3}+1$ power is conformal with the abelian group of type $(m-2, 1, 1)$. Its commutator subgroup is the group of order 2 contained in A. The group G_4 which transforms each operator of H into its $2^{m-3}-1$ power is composed of H and 2^{m-2} operators of each of the orders 2 and 4. These four groups are clearly distinct since no two of them are conformal.

The group G_5 which involves an invariant operator of order 2^{m-2} is also conformal with the abelian group of type $(m-2, 1, 1)$. It cannot be simply isomorphic with G_3 since the latter does not involve any invariant operator of order 2^{m-2}. Moreover, the operators whose orders divide 2^{m-3} constitute an abelian subgroup of G_3, while this is not the case with the operators of G_5. The remaining group G_6 transforms each operator of A into its inverse while it transforms the operators of $H-A$ into

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* *Transactions of the Amer. Math. Society*, vol. 2 (1901), p. 260.

their inverses multiplied by the operator of order two contained in A. It is conformal with G_4, but contains invariant operators of order 4, while G_4 does not have this property. It may also be distinguished from G_4 by the fact that it transforms operators of order 2^{m-2} into their inverses. Hence *there are just six groups in which H is abelian and G/A is non-cyclic.*

The necessary and sufficient condition that G/A is cyclic is that t^2 is in $H - A$. The group G_7 which transforms each operator of H into its inverse is conformal with G_2. The squares of the operators in $G_7 - H$ are not powers of operators of higher order as is the case with the operators of $G_2 - H$. The group G_8 which transforms each operator of H into its $2^{m-3} + 1$ power is conformal with the abelian group of type $(m - 2, 2)$. Its commutator subgroup is of order 2 and all the operators of H whose orders divide 2^{m-3} are invariant. Its group of cogredient isomorphisms is the four group. The group G_9 which transforms each operator of H into its $2^{m-3} - 1$ power is conformal with G_2 and G_7. It is not identical with G_2 since the square of the operators in $G_9 - H$ are not powers of operators of higher order. It is not identical with G_7, since it does not transform its operators of highest order into their inverses, and also because the squares of the operators in $G_9 - H$ give two distinct operators of order two.

Only one case remains, viz., when G transforms the operators of A into their inverses and the operators of $H - A$ into their inverses multiplied by the operator of order 2 contained in A. In this case there is a group G_{10} which contains only operators of order 8 in addition to H. This is the group given by Hallet. It may be defined as the group generated by two operators of order 8 having a common square and such that the product of one of these operators into the inverse of the other is of order 2^{m-2} and generates the fourth power of one of the two generators of order 8. As all the possible cases have now been considered *there are just four groups in which H is abelian and G/A is cyclic.* The groups G_1, G_2, $\cdots$, G_{10} of the present list are simply isomorphic respectively to G_5, G_6, G_1, G_3, G_{10}, G_{15}, G_7, G_2, G_4, G_{16} of the *Transactions* list mentioned above.

§ 2. Groups in which H is non-abelian.

We shall again begin with the case when G/A is non-cyclic. As such a G contains three non-abelian subgroups of order

2^{m-1} which include A, it must transform A according to the three different operators of order 2 in its group of isomorphisms. Hence we may assume without loss of generality that H is conformal with the abelian group of type $(m-2, 1)$. Furthermore, it may be assumed that t transforms each operator of A into its inverse. As t^2 is in A, t is either of order 2 or of order 4. In the former case the group G_{11} generated by H and t contains 3.2^{m-2} operators of order 2 and 2^{m-2} of order 4 in addition to H, and in the latter group G_{12} the number of additional operators of order 4 is 3.2^{m-2} while the number of those of order 2 is 2^{m-2}.

It is easy to see that these are the only groups for which G/A is non-cyclic. Such a G transforms H according to the subgroup of order 8, containing only operators of order 2 besides the identity, in the group of isomorphisms of H. The subgroup I_2 of this group of isomorphisms which transforms A into itself is the direct product of the group of isomorphisms of A and an operator of order 2. Hence it contains only seven operators of order 2 and the way in which t transforms H may be regarded as determined in the above manner. As neither of these groups contains an abelian subgroup of order 2^{m-1} they are distinct from those determined in the preceding section. It remains to consider the case when G/A is cyclic.

As the H of such a G transforms A according to the square of an operator in its group of isomorphisms, it is the same group as in the preceding case. Since t^2 is in $H-A$, t is commutative with some of the operators of $H-A$. Moreover, if t transforms H according to one of the operators of order 4 in I_2 the operators in $G-H$ transform H according to four distinct operators of order 4 in I_2, since the group of cogredient isomorphisms of H is of order 4. As I_2 contains only eight operators of order 4, it is necessary to consider only two cases. In one of these t is commutative with every operator of H whose order divides 2^{m-4}. The resulting group G_{13} is conformal with G_8, but cannot be identical with this group since it contains no abelian subgroup of order 2^{m-1}. The commutator subgroup of G_{13} is the cyclic group of order 4 contained in A.

In the remaining case it may be assumed that t transforms into its inverse each operator of H whose order divides 2^{m-4} while it transforms the operators of highest order in H into their inverses multiplied by an operator of order four contained in A. The resulting group G_{14} contains 2^{m-2} operators of each

of the orders 4 and 8.   Hence, *there are just four groups in which H is necessarily non-abelian.*   In two of these G/A is cyclic while the quotient group is non-cyclic in the other two.

The total number of the non-abelian groups of order 2^m which contain an invariant cyclic subgroup of order 2^{m-2}, but no such subgroup of order 2^{m-1} is therefore fourteen.   The last four were explicitly excluded from my list of these groups which do not contain an abelian subgroup of order 2^{m-1} including A * since Burnside had considered this subject.   Knowing that Burnside gave the correct number of these groups I failed to observe the compensating errors.   It may be added that the title of Hallet's paper as given in both reports noted above is misleading, since every possible group of order 2^m contains an invariant subgroup of order 2^{m-2}.

STANFORD UNIVERSITY,
    March, 1905.

GALILEO AND THE MODERN CONCEPT OF INFINITY.

BY DR. EDWARD KASNER.

THE definition of an infinite assemblage, as one in which a part exists which may be put into one-to-one correspondence with the whole, recognized as fundamental in recent discussions by mathematicians and philosophers, is usually associated with the names of Bolzano, Cantor, and Dedekind.   The object of this note is to call attention to a passage from Galileo which is of significance in this connection.

The passage in question appears incidentally in the work which contains Galileo's most permanent contribution to science, the foundations of dynamics; namely, the Discorsi e dimonstrazioni matematiche of 1638,† often referred to as the

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* *Transactions of the Amer. Math. Society,* vol. 3 (1902), p. 385.

† The full title, taken from a copy of the first edition in the Columbia University library, is as follows :

Discorsi | e | Dimostrazioni | matematiche, | intorno à due nuoue scienze | Attenenti alla | Mecanica & i Movimenti Locali, | del Signor | Galileo Galilei Linceo, | Filosofo e Matematico primario del Serenissimo | Grand Duca di Toscana. | Con vna Appendice del centro di grauità d'alcuni Solidi. | In Leida, | Appresso gli Elsevirii M.D.C. XXXVIII.