relieved, as by sinking a vacant shaft down to it, a violent explosion would follow.

The author concluded from Laplace's law of density that the planet Venus has a rigidity equal to that of a corresponding globe of glass, and that the nuclei of all large bodies are effectively of the highest rigidity. The development of pressure as we decend in the globe was shown to be such as to invalidate the conclusions of Lord Kelvin and Professor G. H. Darwin respecting the consolidation of earth by the building up of a solid nucleus from the sinking of the solidified crust.

G. A. MILLER,
Secretary of the Section.

ON THE DEVELOPMENT OF MATHEMATICAL ANALYSIS AND ITS RELATION TO CERTAIN OTHER SCIENCES. *

ADDRESS DELIVERED BEFORE THE SECTION OF ALGEBRA AND ANALYSIS OF THE INTERNATIONAL CONGRESS OF ARTS AND SCIENCES, ST. LOUIS, SEPTEMBER 22, 1904.

BY PROFESSOR EMILE PICARD.

ONE of the objects of such a congress as that in which we are now assembled is to show the connection between the different parts of science taken in its widest sense. Moreover, the promoters of this meeting have insisted that the relations between different branches should be put in evidence. To undertake a study of this kind, the character of which is somewhat indefinite, one must forget that all is in all; as for algebra and analysis alone, a Pythagorean would be dismayed at the extent of his task, remembering the celebrated formula of the school: "Things are numbers." From this point of view, my subject would be inexhaustible. But for excellent reasons I should not attempt so much. Merely glancing at the development of our science through the ages, and particularly in the last century, I hope to be able to characterize sufficiently the

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* Translated, with the author's permission, by Professor M. W. HASKELL.

rôle of mathematical analysis in its relation to certain other sciences.

I.

It would appear natural to begin by speaking of the very idea of whole number; but this subject belongs not only to logic, but to history and psychology, and would involve us in too many discussions. In the investigation of the number concept, unfathomable depths have revealed themselves. Thus, it is a problem still unsolved as between the two forms, cardinal and ordinal, under which the idea of number presents itself, to know which of the two is anterior to the other; that is to say, whether the idea of number properly so called is anterior to that of order, or inversely. In these questions, the geometric logicians appear to pay too little attention to psychology, and to the information furnished us by uncivilized peoples. Perhaps, too, there is no general answer to the question proposed, this answer varying according to race and mental endowment. I have sometimes thought, apropos of this subject, of the distinction between those in whom the auditory or the visual sense predominates, the former adhering to the ordinal theory and the latter to the cardinal theory. But I shall not waste time on this ground so full of snares; I fear that our modern school of logicians has difficulty in coming to an understanding with the ethnographers and biologists; the latter are always dominated by the doctrine of evolution in questions of origin, and more than one of them regards logic as only the résumé of ancestral experience. Mathematicians have even been reproached with laying down as a principle that there is a human intellect somehow exterior to things, and that it has its own logic. Whatever may be thought of this idea, it has proven useful, nay almost indispensable, in the progress of mathematics; and certainly, supposing that it evolved in the course of prehistoric times, this logic of the human intellect was well established at the time of the early geometric schools among the Greeks; and their works would appear to have been its first code, as is expressed by the story of Plato's writing over the entrance to his school: "Let no man enter here who is not a geometer."

Long before the acquisition from the Arabs of the extraordinary word, algebra, which, it appears, expresses the operation by which equalities are reduced to a certain canonical form, the Greeks had employed algebra without knowing it; one cannot even imagine closer relations than those connecting their algebra and their geometry, or rather, one would be at a loss to classify, if there were occasion, their geometric algebra, in which they reasoned not only about numbers, but also about quantities. Among the Greeks we find also a geometric arithmetic, and one of the most interesting phases of its development is the conflict which arose among the Pythagoreans on this subject between number and quantity apropos of irrationals. While the Greeks cultivated the abstract study of numbers, under the name of arithmetic, their speculative spirit showed less taste for practical calculation, which they called logistic. In the earliest times, the Egyptians and Chaldees, and later the Hindoos and Arabs developed the science of computation. They had been led to it by practical necessities; logistic preceded arithmetic as surveying and geodesy paved the way for geometry; in the same way, moreover, trigonometry was developed by the increasing needs of astronomy. The history of science in its beginnings shows a close relation between pure mathematics and applied mathematics; we shall find this relation again and again as we proceed.

We have remained so far in the domain called in ordinary language elementary algebra and arithmetic; but as soon as the incommensurability of certain quantities was recognized, the notion of infinity made its appearance, and after the time of the paradoxes of Zeno on the impossibility of movement, the summation of geometric progressions could not be avoided. The methods of exhaustion found in Eudoxus and in Euclid properly belong to the integral calculus, and Archimedes calculates definite integrals. Mechanics appears also in his treatise on the quadrature of the parabola, for he finds the surface of the segment bounded by a parabolic arc and its chord at first by depending on the theorem of moments; it is the first example of the relations between mechanics and analysis whose development has been continuous since that time. The infinitesimal method of the Greek geometricians on the measure of volumes has raised questions whose interest is not exhausted even to-day. In plane geometry, two equivalent polygons are equal by addition or subtraction, that is to say, can be decomposed into equal triangles, or be regarded as differences of polygons which can be so decomposed. It is not the same for the geometry of space, and we have recently learned that the equivalence of solid polyhedra, unlike that of plane polygons, can only be established by recourse to the method of exhaustions or to that of limits, which require the axiom of continuity, or the axiom of Archimedes. Without laying any more stress on this point, this glance cast at antiquity shows how closely algebra, arithmetic, geometry, and the first attempts in integral calculus and in mechanics were intermingled at that time, so closely indeed that it is impossible to relate their history separately.

During the Middle Ages and the Renaissance the geometric algebra of the ancients is separated from geometry. Little by little algebra properly so called arrives at autonomy, with its symbolism and its notation more and more perfected; thus is created this language of a wonderful clearness, which provides a true economy of thought and renders further progress possible. This is the moment when distinct divisions are organized. Trigonometry, which in antiquity had been only an auxiliary of astronomy, is developed into an independent subject; at about the same time logarithms made their appearance, and essential elements were thus put in evidence.

II.

In the seventeenth century the analytic geometry of Descartes, different from what I have just called the geometric algebra of the Greeks in the general and systematic ideas upon which it is founded, and the newly-born science of dynamics, were the origin of very great advances in analysis. When Galileo, starting from the hypothesis that the velocity of heavy bodies in their fall is proportional to the time, deduced from this the law of the distance traversed, and verified it afterwards by experiment, he took up again the path where Archimedes had traveled long before, and upon which he was to be followed by Cavalieri, Fermat, and others still down to the time of Newton and Leibniz. The integral calculus of the Greek geometricians was born again in the kinematics of the great Florentine physicist. As to the calculus of derivatives or of differentials, it was stated with precision apropos of the direction of tangents. In reality the origin of the notion of derivatives is in the vague feeling of the mobility of things, and of the greater or less speed with which phenomena take place; this is well expressed by the terms fluent and fluxion, which were used by Newton and which we may believe were borrowed from the ancient mathematician Heraclitus.

The points of view taken by the founders of the science of motion, Galileo, Huyghens, and Newton, had an enormous influence on the orientation of mathematical analysis. In Galileo it was a stroke of genius to perceive that in natural phenomena the determining circumstances of the movement produce accelerations; this led necessarily to the formulation of the principle that the speed with which the dynamic state of a system changes depends in a definite way upon its static condition alone. In a more general way, it came to be postulated that infinitesimal changes, no matter what their nature, which occur in a system of bodies depend solely on the present state of the system. In how far are the exceptions apparent or real? This is a question which was only raised later and which I waive for the moment. From among the principles enunciated an important point for the analyst presents itself; phenomena are governed by differential equations, equations which can be formed when observation and experiment have made known certain physical laws for each category of phenomena. We understand the unbounded hopes to which these results must have given rise. As Bertrand says in the preface to his Traité, "The first successes were such that one might suppose all the difficulties of science overcome in advance, and believe that the mathematician, without being longer occupied in the elaboration of pure mathematics, could turn his thoughts exclusively to the study of natural laws." This was taking for granted that the problems of analysis presenting themselves would offer no very serious difficulties. In spite of the disillusions which the future had in store, there remained this important point, that the problems had taken a precise form, and that the difficulties yet to be overcome could be classified. This constituted an immense advance, one of the greatest which the human mind has ever made. We observe also the reason why the theory of differential equations was destined to acquire a great importance.

I have anticipated somewhat in presenting things under so analytic a form. Geometry was involved in all of these advances. Huyghens, for instance, preferred to follow the ancients, and his Horologium oscillatorium depends both on infinitesimal geometry and on mechanics; likewise, the methods followed in Newton's Principia are synthetic. It is primarily with Leibniz that science begins to follow the paths which were to lead to what we call mathematical analysis; he it is who for the first time, in the later years of the seventeenth century, pronounces the word function. Through his systematic mind, through the numerous problems treated by him, as well as by his pupils, James and John Bernoulli, he established in a definite way the power of methods, to the building up of which a long series of thinkers had contributed, one after another, ever since the remote times of Eudoxus and Archimedes.

The eighteenth century shows the extreme fruitfulness of this new method. It was a curious time, that, of those mathematical duels, when geometers hurled defiances at each other, struggles which were not always without bitterness, when Leibnizians and Newtonians met in the arena. From the purely analytic point of view, the classification and study of simple functions is particularly interesting; the idea of function upon which analysis rests is thus developed little by little. The celebrated works of Euler occupied an important place at that time. However, the numerous problems which were presented to mathematicians hardly left them time to scrutinize the principles; the theoretical foundations themselves are slowly cleared up, and the saying attributed to d'Alembert: "Go forward and faith will come to you," is thoroughly characteristic of this epoch. Of all the problems raised at the end of the seventeenth century or during the first half of the eighteenth century, it is sufficient for me to recall those isoperimetric problems which were to give birth to the calculus of variations. I prefer to emphasize the more intimate relations still between analysis and mechanics, when after the inductive period of the early years of dynamics the deductive period arrived, in which the effort was made to give a definitive form to the principles. The mathematical and formal development then played the essential rôle, and the analytic language was indispensable to the greatest extension of these principles. There are moments in the history of sciences, and perhaps of societies, in which the mind is sustained and carried forward by the words and symbols which it has created, and in which generalizations present themselves with the least effort. Such was particularly the rôle of analysis in the formal development of mechanics. Let me here be permitted one remark. It is often said that an equation contains only what has been put into it. It is easy to reply in the first place that the new form under which things are found often constitutes by itself alone an important discovery. But there is something more: analysis, by the simple play of its symbols, may suggest generalizations far beyond the original limits. Is it not so with the principle of virtual velocities, of which the first idea came from the simplest mechanisms? The analytic form into which it is translated suggested extensions which led far from the point of departure. Similarly, it is not right to say that analysis has created nothing, since these more general conceptions are its work. Another example is furnished us by the system of Lagrange's equations; here calculus transformations have given the type of differential equations to which one tends to refer to-day the notion of mechanical explanation. There are in science few examples comparable with this of the importance of the form of an analytic relation and of the power of generalization of which it can be capable. It is very clear that in every case the generalizations suggested must be stated precisely by an appeal to observation and experiment; finally, it is still the calculus which will deduce the remote consequences to be submitted to the same tests, but this is an order of ideas which I am not to consider here.

Under the impulse of the problems suggested by geometry, mechanics and physics, we see almost all of the great divisions of analysis develop or originate. In the first place we met equations with a single independent variable. Soon equations with partial derivatives were to appear in connection with vibrating strings, the mechanics of fluids and the infinitesimal geometry of surfaces. This was all a new analytic world; the very origin of the problems treated was a help to prevent our going astray in the first steps, and in the hands of Monge geometry rendered useful service to the newly born theories. But of all the applications of analysis, no one was more brilliant than the problems of celestial mechanics suggested by the knowledge of the laws of gravitation, with which are associated the names of the greatest mathematicians. Theory never had a finer triumph; perhaps one might add that it was too complete, for this is the very moment above all in which were conceived for natural philosophy those hopes, premature to say the least, of which I spoke above. In all this period, especially in the second half of the eighteenth century, what strikes us with admiration, and at the same time introduces some confusion, is the extreme importance of the applications realized, while the pure theory appeared still so insecure. We see this when certain questions are raised like that of the degree of arbitrariness in the integral of vibrating strings, which gives rise to a protracted discussion still far from being concluded. Lagrange felt these insufficiencies when he published his theory of analytic functions, in which he endeavors to give a precise foundation to analysis. We cannot too much admire the marvellous presentiments which he had of the part to be played by the functions which we call, as he did, analytic; but we may admit our amazement at the demonstration which he thought he had given of the possibility of the development of a function in Taylor's series. The requirements in the questions of pure analysis were less at this epoch. Trusting to intuition, mathematicians took up with certain plausibilities and depended implicitly upon certain hypotheses which it appeared to them useless to formulate explicitly; at bottom, they trusted in the solidity of ideas which had been so many times proved fruitful; this is pretty nearly the well known saying of d'Alembert. The need of rigor in mathematics has had its successive approximations, and in this respect our science has not the absolute character so many persons attribute to it.

III.

We have now arrived at the first years of the nineteenth century. We have explained how the great majority of the analytic investigations in the eighteenth century were occasioned by geometric, and particularly by mechanical and physical problems; and we have found scarcely a trace of the logical and æsthetic predispositions which have given such a different physiognomy to so many mathematical works, especially of the last two-thirds of the nineteenth century. Let us not anticipate, however. After so numerous examples of the influence of physics on the development of analysis, we shall meet still others, and those of the utmost importance, in Fourier's theory of heat.

Fourier begins by forming the partial differential equations which govern the temperature. What are the boundary conditions which allow the determination of a solution of a partial differential equation? With Fourier, the conditions are suggested by the physical problem, and the methods which he followed served as models for the geometric physicists of the first half of the last century. One of them consists in forming series of certain simple solutions. Fourier obtained in this way the first types of more general developments than the trigonometric; as in the problem of the cooling of a sphere, where he applies his theory to the terrestial globe, and investigates the law governing the variations of temperature in the sun, endeavoring to obtain numerical applications. In the face of so many beautiful results, we understand the enthusiasm which glows in every line of Fourier's introductory chapter. Speaking of mathematical analysis, "There cannot be," says he, "a language more universal and more simple, more devoid of errors and of obscurities, that is to say more suitable for the expression of the invariable relations of natural objects. Considered from this point of view, it is as wide as nature itself; it defines all perceptible relations, measures time and space, forces and temperatures; this difficult science is formed slowly, but it maintains all the principles that it has once acquired. It increases and strengthens without ceasing in the midst of so many errors of the human intellect." The eulogy is magnificent; but between the lines we read the tendency to regard analysis solely as the helpmeet, however incomparable, of natural science, a tendency in conformity, as we have seen, with the development of science during the two centuries preceding; but we are just coming to the period in which new tendencies were to make their appearance. Poisson having recalled, in a report on the *Fundamenta*, Fourier's complaint about Abel and Jacobi that they were not primarily interested in the movement of heat, Jacobi writes to Legendre: "It is true that M. Fourier was of the opinion that the chief end of mathematics was public utility and the explanation of natural phenomena; but a philosopher like him ought to have known that the sole end of science is the honor of the human intellect, and that under this head a problem of number is as important as a problem of the system of the world." This was also without doubt the opinion of the great mathematician of Göttingen, who called mathematics the queen of sciences and arithmetic the queen of mathematics. It would be absurd to put these two tendencies in opposition to each other; the harmony of our science lies in their synthesis. The moment had to come when the necessity would be felt of inspecting the foundations of the structure and of making an inventory of the accumulated stores of wealth by the introduction of more of the critical spirit. Mathematical thought was to acquire more strength by doubling on its track; problems become exhausted for a time, and it is not good when all the investigators remain in the same road. Besides, unexplained difficulties and paradoxes made advances in the pure theory necessary. The road along which it must move was indicated in a general way, and therein it could proceed independently without necessarily losing contact with the problems proposed by geometry, mechanics, and physics. At the same time, more interest was going to be attached to the philosophic and artistic side of mathematics, through a trust in a sort of preëstablished harmony between our logical and æsthetic satisfactions and the needs of future applications.

Let us briefly call to mind some points in the history of the revision of the principles in which Gauss, Cauchy, and also Abel were the pioneers.

Precise definitions for continuous functions and their most immediate properties and simple rules for the convergency of series were formulated, and soon the possibility of trigonometric developments was established under very general conditions, thus justifying Fourier's boldness. Certain geometric intuitions respecting areas and arcs gave place to rigorous demonstrations. The geometricians of the eighteenth century necessarily tried to take account of the degree of generality in the solution of ordinary differential equations. The comparison with difference-equations easily led to the result, but this kind of proof will not stand a close inspection. Lagrange, in his lessons on the calculus of functions, introduced more precision, and starting from Taylor's series, he saw that an equation of order m leaves undetermined the function and its first $m - 1$ derivatives for the initial value of the variable; we are not surprised that Lagrange had not considered the question of convergency. In twenty or thirty years the requirements in the rigor of proofs had increased. We know that the two preceding modes of proof are susceptible of all the precision necessary. With the first, no new principle was needed; with the second, the theory had to be developed in a new way.

Up to this point, functions and variables had remained real. By the consideration of complex variables the field of analysis was extended. The functions of a complex variable with a unique derivative are necessarily developable in a Taylor's series; we thus fall back on the method of development whose interest was understood by the author of the theory of analytic functions, but whose importance could not be fully exhibited so long as the investigation was limited to real variables. Moreover, they owe the great part which they still continue to play to the facility with which they can be handled, and to their convenience in calculation. The general theorems in the theory of analytic functions have permitted the giving of an exact reply to questions previously undecided, like the degree of generality of the integrals of differential equations. It became possible to complete the demonstration sketched by Lagrange for an ordinary differential equation; and for the case of a partial differential equation, or of a system of such equations, precise theorems have been established. Not that the results obtained on this last point, important as they are, solve completely the different questions which can be brought forward; for in mathematical physics, and even in geometry, the boundary conditions admit such a variety of forms that the problem called by Cauchy's name often appears in quite a narrow form. I shall return to this important point presently.

IV.

Without adhering to the historical order, let us return to the development of mathematical physics in the last century, in so far as it concerns analysis. The problems of thermal equilibrium led to the equation already encountered by Laplace in the study of attraction. Few questions have been the subject of so much labor as this famous equation. The boundary conditions may be of various forms. The simplest case is that of the thermal equilibrium of a body whose surface elements are constant — the temperatures given; from the physical point of view, it may be regarded as evident that the temperature, being continuous in the interior, since there is no source of heat, is determined when it is given at the surface. A more general case is that in which, the condition remaining permanent, there would be radiation outward with an intensity varying at the surface according to a given law; in particular, the temperature may be given over one portion of the surface while there is radiation over the remainder. These questions, which have not yet been solved in the most general case, have contributed enormously to the orientation of the theory of partial differential equations. They have called attention to types of determination of the integrals which would not have presented themselves if the point of view had remained purely abstract. Laplace's equation had already been met in hydrodynamics, and in the study of attraction in the inverse ratio of the square of the distance. This last theory has led to the putting in evidence of the most essential elements, like the potential of simple layers and of double layers. There have appeared in this theory analytic combinations of the utmost importance, which have since been generalized to a notable degree, as for instance, Green's formula.

The fundamental problems of electrostatics belong to the same order of ideas, and it was surely a fine triumph for the theory when the discovery was made of the celebrated theorem on electric phenomena in the interior of hollow conductors, which Faraday rediscovered by experiment without being acquainted with Green's paper. This whole magnificent aggregation became the model for the theories established in mathematical physics, which seem to us to have almost reached perfection, and which exert such a happy influence on the progress of pure analysis by suggesting to it the most beautiful problems.

The theory of functions will afford us another remarkable comparison. The analytic transformations called into play are no different from those already encountered in the steady movement of heat. Certain fundamental problems in the theory of functions of a complex variable have accordingly been able to exchange their abstract enunciation for a physical form, like that of the distribution of the temperature on a closed surface of any connectivity whatever and without radiation, in thermal equilibrium, with two sources of heat necessarily corresponding to equal and opposite flows. Interpreting this geometrically, we find ourselves face to face with a question respecting abelian integrals of the third kind in the theory of algebraic curves.

The preceding examples, in which we have briefly considered only the equations of heat and of attraction, show that the influence of the physical theories is exerted not only on the general nature of the problems to be solved, but even in the details of the analytic transformations. Thus in recent memoirs on partial differential equations, it is customary to denote by the name of Green's formula one which was suggested by the original formula of the English physicist. The theory of dynamic electricity, and that of magnetism, in the hands of Ampère and Gauss were the source of notable advances; the study of curvilinear integrals and of surface integrals owe all their development to this theory, and formulas like that of Stokes, which might also be called Ampère's formula, appeared for the first time in memoirs on physics. The equations of the propagation of electricity, with which are connected the names of Ohm and Kirchhoff, while presenting a strong analogy with those of heat, are often subject to slightly different boundary conditions; we know how much cable telegraphy owes to the thorough discussion of the integrals of an equation of Fourier's applied to electricity. The equations of hydrodynamics formulated long ago, the equations in the theory of elasticity, those of Maxwell and of Hertz in electro-magnetism, have presented problems analogous to those mentioned above, but under still more varied conditions. Unsurmounted difficulties are met there, it is true, but how many beautiful results are due to the study of particular cases, the number of which we should like to see increased. We should mention also, as of both analytical and physical importance, the profound differences exhibited by propagation according to the phenomena studied. With equations like those of sound there is propagation by waves; with the equation of heat every variation is felt instantly at every distance, but very slightly at very great distances, and therefore it is impossible to speak of a velocity of propagation. In other cases, of which Kirchhoff's equation relative to the propagation of electricity with induction and capacity offers the simplest type, there is a wave-front with a definite velocity, but with a residue behind which is not obliterated. These different circumstances reveal very different properties of the integrals; they have been thoroughly studied only in a small number of particular cases, and these investigations raise problems involving the profoundest ideas in modern analysis.

V.

I shall enter into some analytic details of especial interest to mathematical physics. The question of the generality of the solution of a partial differential equation has presented some apparent paradoxes. For one and the same equation the number of arbitrary functions was not always the same, depending upon the form of the integral considered. Thus Fourier, studying the equation of heat in an unlimited medium, regards it as evident that a solution will be determined if its value is given for $t = 0$, that is to say, as an arbitrary function of the three coördinates, x, y, z; from Cauchy's point of view, on the other hand, it might be considered that the general solution involves two arbitrary functions of three variables. At bottom, the question, in the way in which it was stated for a long time, has no precise significance. In the first place, if only analytic functions are considered, any finite number of functions whatever, with any number of independent variables, exhibit from the arithmetical point of view no more generality than a single function of a single variable, since in both cases the totality of the coëfficients in the development forms an enumerable sequence. But there is something more. In fact, besides the conditions expressed by given functions, an integral is subject to conditions of continuity, or must become infinite in a definite way for certain elements; we may then be led to regard the condition of continuity in a given space as equivalent to an arbitrary function, and it is then quite clear in how far the question of the enumeration of the arbitrary functions is badly stated. It is often a delicate matter to prove what conditions determine a solution uniquely, unless one is willing to be satisfied with plausibilities; it is then necessary to state precisely the manner in which the function and certain of its derivatives behave. For instance, in Fourier's problem regarding an indefinite medium, certain hypotheses must be made about the function and its first derivatives at infinity if we wish to show that the solution is unique. Formulas analogous to those of Green are of great service, but the proofs deduced from them are not always absolutely rigorous when we suppose boundary conditions satisfied which are, *a priori* at least, not necessary. This is only another example of the increased demand for rigor in proofs. Let us observe, besides, that the new investigations rendered necessary have often led to a better understanding of the nature of integrals; this is because true rigor is productive, being distinguished in this from another rigor which is purely formal and tiresome, casting a shadow over the problems which it touches.

The difficulties in the proof of the uniqueness of a solution may differ very much according to whether all the integrals of the equations considered are analytic or not. There is an important point here, which shows that it is sometimes necessary to reckon with non-analytic functions, even when we should prefer to avoid them. For example, it cannot be affirmed that Cauchy's problem determines a solution uniquely when the data of the problem are general, that is to say, are not characteristic. This is surely the case when only analytic integrals are considered; but in the case of non-analytic integrals there may be contacts of an infinite order, and in this case the theory is not in advance of the applications; the contrary is true, as is shown by the following example. Does the famous theorem of Lagrange on the velocity-potential in a perfect fluid hold for a viscous fluid? It has been possible to give examples where the coördinates of the different points of a viscous fluid starting from rest are not expressible as analytic functions of the time starting from the beginning of the movement, and where the rotations, as well as all their derivatives with respect to the time, are zero at that instant, and yet are not identically zero; in this case Lagrange's theorem does not hold. These considerations are sufficient to show the interest there may be in being assured that all the integrals of a system of partial differential equations, continuous and with continuous derivatives up to a definite order in a certain field of real variables, are analytic functions, supposing, of course, that the equations themselves contain only analytic elements. For linear equations there are precise theorems, all the integrals being analytic if the characteristics are imaginary, and very general propositions have also been obtained in other cases. The boundary conditions which we are led to set up are very different according to whether the integrals of the equation in question are analytic or not. A case of the former type is given by the generalized problem of Dirichlet. In this case, an essential part is played by the conditions of continuity, and in general, the solution cannot be extended on the two sides of the continuum with reference to which the data are defined. It is quite different in the second case, where the relation of this continuum to the characteristics is of principal importance, and where the field of existence of the solution is presented under entirely different conditions. All these notions, difficult to state precisely in ordinary language, are fundamental in mathematical physics, and of not less interest in infinitesimal geometry. It will be sufficient to recall the fact that all surfaces of constant positive curvature are analytic, while there are surfaces of constant negative curvature which are non-analytic.

Since the time of the ancients, the vague belief in a certain economy in natural phenomena has made its way; one of the first exact illustrations is furnished by Fermat's principle with regard to the economy of time in the transmission of light. Afterwards it came to be recognized that the general equations of mechanics correspond to a problem of minimum, or more exactly of variation, and in this way there was obtained the principle of virtual velocities, then Hamilton's principle, and the principle of least action. A large number of problems were thus shown to correspond to the minima of certain definite integrals. This amounted to a very important step in advance, for in many cases the existence of a minimum could be considered self-evident, and in consequence the proof of the existence of a solution was accomplished. This method of reasoning has been of immense service; the greatest geometricians have taken up with it, Gauss in the problem of distribution of an attracting mass corresponding to a given potential, Riemann in his theory of abelian functions. In the present day, our attention has been called to the dangers of this species of demonstration; it is possible that the minima are simply limits that cannot be really attained by actual functions which possess the necessary properties of continuity. We are no longer contented with the plausibilities presented by the method of reasoning which has long been classic. Whether we proceed indirectly, or whether we endeavor to give a rigorous proof of the existence of a function corresponding to the minimum, the road is long and arduous. In other respects, it will be none the less always advantageous to connect a question of mechanics or of mathematical physics with a problem of minimum; in the first place, this affords a fruitful source of analytic transformations, and besides, in the very processes of the investigations of the variations, useful suggestions with regard to the boundary conditions may present themselves; a beautiful example was furnished by Kirchhoff in the delicate investigation of the boundary conditions in the equilibrium of flexion of plates.

VI.

I have been led to dwell particularly on partial differential equations. Examples chosen from rational mechanics and from celestial mechanics would easily show the part played by ordinary differential equations in the progress of these sciences, whose history, as we have seen, has been so closely united with that of analysis. When the hope of integrating by means of simple functions is given up, an effort is made to find developments allowing us to follow a phenomenon as long as possible, or at least to obtain hints as to its qualitative behavior. On the practical side, the methods of approximation form a very important part of mathematics, and thus the most advanced parts of arithmetic are related to the applied sciences. As for series, they occur, to begin with, in the very existence-proofs of the integrals; for instance, Cauchy's first method gives convergent developments as long as the integrals and their differential coefficients are continuous. When any circumstance allows us to predict that this is always the case, we obtain developments always convergent. In the problem of n bodies, we can in this way obtain developments which hold so long as there are no impacts. If the bodies repelled each other instead of attracting, there would be no danger of this happening, and we should obtain developments which would hold indefinitely; unfortunately, however, as Fresnel once said to Laplace, "Nature is not concerned about analytic difficulties," and the heavenly bodies attract instead of repelling each other. We might even be tempted to go further than the great physicist, and say that nature has strewn the path of the analyst with difficulties. Thus, to take another illustration, when a system of differential equations of the first order is given, we can usually decide whether the general solution is stable in the neighborhood of a point or not; and in order to find developments in series which hold for the stable solutions, it is only necessary that certain inequalities be satisfied.

But if we apply these results to the discussion of stability in the equations of dynamics, we find ourselves exactly in the particular unfavorable case. In general, in this very case, it is impossible to make a definite statement with regard to the stability; in the case where a force-function has a maximum, a standard, but indirect, method of reasoning establishes the stability, which cannot be deduced from any development that holds good for every value of the time. Let us not regret these difficulties; they will be the source of future progress. Such also are the difficulties still presented to us, in spite of the great amount of labor expended upon them, by the equations of celestial mechanics; the astronomers have deduced from them, since the time of Newton, by means of practically convergent series and of happily conducted approximations, pretty nearly all that is necessary for the prediction of the movements of the celestial bodies. The analysts would demand more, but they hardly hope any longer to perform the integration by means of simple functions or of developments which are always convergent. What has been best learned from some recent remarkable investigations, is the immense difficulty of the problem; a new road has been opened, however, by the study of particular solutions, like periodic solutions and asymptotic solutions, which have already been put into use. It is perhaps not so much for the sake of practical necessities as for the sake of not admitting defeat, that analysis will never reconcile itself to abandoning without a positive victory a subject which has been the scene of so many brilliant triumphs; and, besides, what finer field could be found to test the strength of the newly born or rejuvenated theories of the modern doctrine of functions than this classic problem of n bodies?

The analyst is rejoiced to meet in the applications equations which he can integrate with known functions, with transcendentals already classified. Such meetings are unfortunately rare; the pendulum problem, the classic cases of the motion of a solid body about a fixed point, are examples where the integration can be performed by the use of elliptic functions. It would also be extremely interesting to meet a question in mechanics which could be the origin of an important discovery in the theory of functions, such as the discovery of a new transcendental enjoying some remarkable property; I should be at a loss to give an illustration, unless I were to refer the origin of the theory of elliptic functions to the pendulum. The relations between theory and application are here much less intimate than in the questions of mathematical physics considered above. This explains the fact that, during the last forty years, the study of the ordinary differential equations connected with analytic functions has had for the most part a totally abstract theoretical character. The pure theory has been notably in the lead; we have had occasion to say that it was good that it was so, but here there is evidently a question of degree, and one could wish to see some old problem profit by the progress that has been made. We should not be especially at a loss to give some examples in this connection, and I will call to mind only those linear differential equations involving arbitrary parameters whose singular values are roots of transcendental integral functions, and that one in particular which establishes a correspondence between the successive harmonics of a vibrating membrane and the poles of a meromorphic function.

It occurs also that the theory may be an element of classification by leading to the investigation of the conditions under which the solution belongs to a definite type, as for instance, when the integral is a uniform function. There have been and there will be again plenty of interesting discoveries in this field. The case of the movement of a heavy solid body, treated by Mme. de Kowalewski, where abelian functions were employed is a remarkable instance.

VII.

In studying the mutual relations between analysis and mechanics and mathematical physics, we have more than once found the road opened by infinitesimal geometry, which has proposed so many celebrated problems; in many difficult questions, a happy combination of the calculus with synthetic methods has been the means of great advances, as is shown by the theories of applicable surfaces and of triply orthogonal systems. There is another branch of geometry which is of great importance in some analytical investigations; I mean the geometry of situation, or analysis situs. We know that Riemann made from this point of view a complete study of the continuum of two dimensions, upon which rests his theory of algebraic functions and their integrals. When this number of dimensions is increased, the questions of analysis situs become necessarily more complicated; geometric intuition ceases, the study becomes purely analytic, and the thought is guided only by analogies which may be deceptive and require close investigation. The theory of algebraic functions of two variables, which transports us into a space of four dimensions, without reaping so much advantage from analysis situs as does the theory of functions of one variable, owes to it, nevertheless, some useful orientations. Here is still another class of questions which involve the geometry of situation; in the study of curves drawn on a surface and defined by differential equations, the connectivity of this surface plays an important part; this is what occurs in the case of geodesic lines. The notion of connectivity, moreover, presented itself long ago in analysis, when the study of electric currents and of magnetism had led to non-uniform potentials; in a more general manner, certain multiform integrals of some partial differential equations are met in difficult theories like that of diffraction, and there are various investigations to be pursued in this direction.

From a different point of view, I ought to mention also the relations between algebraic analysis and geometry which appear so elegantly in the theory of groups of finite order. A regular polyhedron, an icosahedron for example, is on the one hand the well known solid; it is also for the analyst a group of finite order corresponding to the different ways of bringing the polyhedron into coincidence with itself. The investigation of all the types of groups of movements of finite order interests not only geometricians, but crystallographers as well; it amounts essentially to a study of the groups of ternary linear substitutions of determinant ± 1, and leads to the thirty-two classes of symmetry for the molecular structure of crystals. The corresponding groupings of polyhedra in systems so as to fill up space exhaust all the possibilities in the investigation of the structure of crystals. Since the epoch when the notion of groups was introduced into algebra by Galois, it has undergone in different ways remarkable developments, to such an extent that it is met to-day in every branch of mathematics. In the applications it appears to us above all as an admirable instrument of classification. Whether we consider substitution groups or Sophus Lie's transformation groups, whether we consider algebraic equations or differential equations, this comprehensive theory permits us to take account of the degree of difficulty of the problems treated and teaches us to employ the special circumstances which present themselves in each; for this reason it must prove as essential to mechanics and to mathematical physics as it is to pure analysis.

Mechanics and mathematical physics have been developed to the point where a mathematical form may be given to almost all their theories; certain hypotheses and the knowledge of elementary laws have led to the differential relations which constitute, at least for the time, the final form of statement of these theories. Little by little they have seen their field enlarged by the principles of thermodynamics; and to-day chemistry is tending, in its turn, to take on a mathematical form. I shall call to witness only the celebrated memoir by Gibbs on the equilibrium of chemical systems, of a character so analytic that it requires some effort for chemists to recognize laws of great importance under their algebraic cloak.

It seems that chemistry has now emerged from the pre-mathematical period in which every science takes its start, and that a day must come when vast theories will be established, analogous to those of our present mathematical physics, but of a much wider scope, and including all the physico-chemical phenomena. It would be premature to inquire whether analysis will find in their developments the source of new advances; we do not even know what analytic types we may encounter. I have constantly spoken of differential equations governing phenomena; will this be always the final form into which a theory is crystallized? I know nothing about it with certainty, but yet we must remember that several hypotheses of a more or less experimental character have been made ; among them is one which may be called the principle of *non-heredity*, which postulates that the future of a system depends only on its present state and on its state at the infinitely near instant, or more briefly, that accelerations depend only on positions and velocities. We know that in certain cases this hypothesis is not admissible, at least with the quantities directly considered ; in this connection a misuse has sometimes been made of the memory of matter, as if it remembered its past, and the life of a piece of steel has been spoken of in moving terms. Various attempts have been made to present a theory of these phenomena where a far-reaching past seems to intervene ; this is not the place to discuss them. An analyst may think that in cases of such complexity it will be necessary to abandon the form of differential equations and to be resigned to the consideration of *functional equations* involving definite integrals which will be the evidence of a sort of heredity. In view of the interest attaching at the present time to functional equations, one might believe in a presentiment of future needs.

VIII.

After speaking of non-heredity, I hardly dare to touch on the question of the applications of analysis to biology. There is no doubt that functional equations of biological phenomena will not be constructed so easily ; * the attempts made hitherto have been confined to a very modest range of ideas. However, the effort is making to come out of a purely qualitative field and to introduce quantitative measurements. In the question of the variation of certain characters, reliance is placed on processes of measurement and on statistical data which are interpreted by curves of frequency. The modifications of these curves with successive generations and their decomposition into distinct curves could give the measure of stability of species or of the rapidity of mutations, and we know what interest attaches to these questions in recent botanical investigations. In all this there is such a large number of parameters, that the question is whether the infinitesimal method itself can be of any service.

* In his article on "Lamarck's principle and the heredity of bodily changes," M. Giard says of heredity : "It is an integral, it is the sum of variations produced in each foregoing generation by the primary factors of evolution." See Controverses transformistes, p. 135.

Some laws of a simple arithmetical character, like those of Mendeleëff, have sometimes renewed confidence in the old aphorism which I quoted at the start, that all things are explained by numbers; but in spite of justifiable hopes, it is clear that, taken as a whole, biology is still far from entering into a really mathematical period.

It is otherwise with political economy, according to certain economists. Following Cournot, the school of Lausanne has made an extremely interesting effort to introduce mathematical analysis into political economy. Under certain hypotheses which agree at least with certain limiting cases, there is found in learned treatises an equation between quantities of merchandise and their prices which recalls the equation of virtual velocities in mechanics; this is the equation of economic equilibrium. In this theory, a function of the quantities plays an essential part not unlike that of the potential function. Moreover, the leading representatives of the school insist on the analogy between economic and mechanical phenomena.

"As rational mechanics" says one of them, "considers material points, so pure economics considers the homo œconomicus." Here also, of course, are found again equations analogous to those of Lagrange, the inevitable pattern of all mechanics. With all admiration for these daring performances, we are inclined to fear that the authors have neglected certain concealed masses, as Helmholtz and Hertz would have said. But whatever happens to them, these theories contain a curious application of mathematics, which has already been of great service, at least in some well circumscribed cases.*

Gentlemen, I have finished this brief history of some of the applications of analysis with the reflections which it has suggested to me from time to time. I have given a résumé far from complete; for instance, I have omitted to speak of the calculus of probabilities, which demands such subtlety of thought, and whose artifices Pascal refused to explain to the Chevalier de Méré because he was not a geometrician. Its practical utility is of the first order, its theoretical interest has always been great, and has been still more increased recently, thanks to the investigations which Maxwell called statistical and which tend to exhibit mechanics in an entirely new light.†

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*See upon this topic: La méthode mathématique en économie politique, by E. Bonoier, and Petite traité d'économie politique mathématique, by H. Laurent.

†See Elementary principles in statistical mechanics, by J. W. Gibbs, and Léçons sur la theorie des gaz, by L. Boltzmann.

I hope, however, to have shown in this sketch the origin and reason of the close bonds which unite analysis with geometry and with physics, more generally with every science bearing on quantities which can be measured numerically. The mutual influence of analysis and the physical theories has been in this respect particularly instructive. What has the future in reserve? More difficult problems, corresponding to a higher order of approximation, will introduce complications that we can only vaguely foresee, speaking, as I did just now, of functional equations replacing systematically the differential equations of our present time, or again of integration of equations infinite in number, and involving an infinity of unknown functions. But whatever happens, mathematical analysis will always remain that language which, in Fourier's words, "has no symbols to express confused thoughts," a language endowed with a wonderful power of transformation and able to condense within its formulas an immense number of results.

ON THE CLASS OF THE SUBSTITUTIONS OF VARIOUS LINEAR GROUPS.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society, April 29, 1905.)

1. In a recent memoir * by M. Edmond Maillet the question of the possible number of real elements of a geometric configuration (such as the 27 straight lines on a cubic surface) is made to depend upon the class of the substitutions of the Galois group G of the equation determining the elements or of any known group containing G. In view of such an application in various geometric and function-theoretic problems, Maillet emphasizes the importance of a knowledge of the class of the substitutions of various linear modular groups. For the general linear group on m variables with coefficients modulo n, Maillet determines completely the class of its substitutions when n is a prime, while for n a power of a prime he determines a set of

* *Annales de l'Université de Toulouse* (2), vol. 6 (1904), pp. 277–349. In a paper to appear in the July number of the *Annals of Mathematics*, I obtain wide generalizations of Maillet's geometric results, the methods employed being much simpler than his.