The linear difference equation is now algebraically integrable. For the characteristic invariants of G_1 being the elements of a fundamental system $[y_x]$, it follows that any rational function, and in particular any symmetric function of the $n\nu$ solutions $[y_x]$, $T_1[y_x]$, $T_2[y_x]$ $\cdots$, $T_{\nu-1}[y_x]$ remains invariant under the permutations of G and is therefore rational in x.

THE UNIVERSITY OF CHICAGO,
April, 1904.

EXPERIMENTAL AND THEORETICAL GEOMETRY.

Experimental and Theoretical Course of Geometry. By A. T. WARREN, M.A. Formerly Scholar of Corpus Christi College, Oxford. Assistant Master at Dover College. Oxford, at the Clarendon Press, 1903.

NOTWITHSTANDING the flood of books that have been put out in recent years under such titles as Experimental, Intuitional, Practical, Observational, Concrete, Heuristic and Objective Geometry, the common aim of which it has been to supply inductive knowledge of space relations, it is generally acknowledged by those who are practically engaged in the teaching of geometry that the ideal text-book is yet to be written. The tendency and danger in the class of books referred to has been to displace demonstrative geometry by offering in its place a pseudo-geometry — a conglomeration of interesting exercises well adapted to furnish the pupil with the facts of geometry, but imparting little if any training in close, consecutive thinking. In their attempt to escape the charges to which Euclid is open as a text-book for beginners, these books go to an opposite extreme and treat geometry as one would a natural science, forgetting that geometry proper is not an experimental science, that its essential object as a branch of study is not the discovery of facts but rather the discerning of relations between ideas. Indeed its very existence as an independent subject of study in the common and secondary schools is conditioned upon the recognition that its function differs from that of every other science, that whatever value may be placed upon its incidental results the one paramount virtue of geometry is that it develops the reasoning powers, just as the natural sciences develop the powers of observation and the judgment.

The book before us seems to be an attempt to avoid extremes. On the one hand it recognizes the need of a thorough grounding in the fundamental conceptions and facts and language of geometry, so that pupils may not fail for a want of adequate conceptions of the things reasoned about. On the other hand, it brings the experimental work within a compass that need not infringe seriously upon the time that should be given to geometry proper — demonstrative geometry.

The little volume contains 248 pages of open composition, of which 111 pages are devoted to experimental, the remaining 137 pages to theoretical geometry.

The experimental course consists of twenty-seven chapters. The general plan of each chapter is as follows :

1. The definitions of the principal terms are stated and carefully explained.

2. The essential properties and the relations of the various parts of the figure are developed inductively by means of the usual devices, most prominent among which are measurement of distances and angles by the use of linear scales and protractor, ruler and compass constructions, the use of tracing paper for transferring magnitudes, paper cutting aud paper folding, rotation of lines and figures, the use of squared paper, construction of cardboard models.

3. A summary of conclusions which have been drawn.

4. A set of questions and exercises involving the concepts and relations that have been developed in the chapter.

To show the author's ingenuity and originality in the treatment of definitions we select two typical illustrations. Euclid's definition of a straight line is stated and explained as follows :

" *A* STRAIGHT *line is a line which lies* EVENLY *between its end points.*

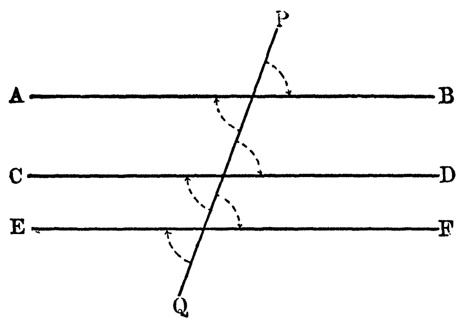
" To understand this definition, draw a line AB right across your paper. Cut the paper along AB; both edges of the paper, where it is cut, will be copies of the line AB. If now you can slide one edge in every possible way right along the other without leaving a space between them, the line AB is said to lie evenly between its end points A and B.

"Apply this test to show that the crease formed by folding a piece of paper is a straight line."

Chapter XVII opens with a definition of parallel lines :

"*Two straight lines are said to be* PARALLEL *when one can be moved* WITHOUT TURNING *so as to lie along the other.*

"Draw a straight line PQ to cut the ruled lines on your squared paper; let AB, CD and EF represent three of them. Over this lay a piece of transparent paper; on this take a tracing of PQ and of one of the straight lines met by PQ.



"Slide the transparent paper so that the tracing of PQ shall slide along PQ. The other traced line will thus move without turning, and it will be seen to coincide in turn with other ruled lines of the squared paper.

"You thus see that those ruled lines of the squared paper, which are not ⊥ to one another are ||.

"It is also evident from the way in which certain ∠s coincide during the movement of the tracing, that the ∠s through which PQ would have to turn, in order to coincide with each of the || lines, are all equal."

It is further pointed out how the mechanical way of constructing parallel lines by sliding a set square along the edge of a T-square, satisfies the given definition of parallel lines.

The experimental methods employed in the discovery of relations are so various that it is difficult to select typical examples. In dealing with angles and angle sums, illustration by rotation of lines is most frequently employed. Thus by turning one of two intersecting lines about their point of intersection until it coincides with the other, it is shown that vertical angles are equal. By revolving one side produced of a triangle successively about the vertices of the triangle, through angles equal respectively to the corresponding angles of the triangle, the conclusion is drawn that the angle sum equals two right angles. Similarly it is shown that the exterior angle sum of any polygon equals four right angles. Again, by counting the number of half revolutions which a line makes in revolving successively through consecutive angles of a polygon, the theorem concerning the interior angle sum of any polygon is derived.

The conversion of triangles and polygons into rectangles and of rectangles into squares and finally the quadrature of the circle is accomplished by paper cutting and the use of squared paper. Similar figures are defined as figures which are reduced or enlarged copies of one another, and their areas are compared by actually tabulating the number of equal elementary areas contained in figures having different linear dimensions.

An admirable feature of the experimental course is that the experiments can all be easily performed by the average pupil and that no apparatus and material is required that is not everywhere at hand.

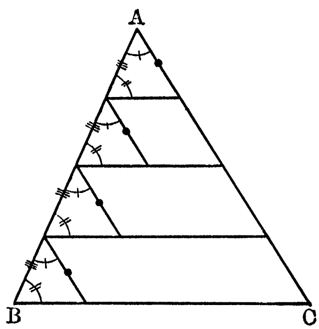
The exercises following each chapter could easily have been made more interesting and instructive. The sole aim seems to have been to give practice in the results already obtained, rather than to suggest new methods and applications. On the whole the impression is left that the exercises have not received the same care that has been bestowed on the body of the text.

Probably the greatest merit of the experimental course as compared with other texts of a similar character is its conciseness and brevity. This part of the work could be completed by a high school class, reciting daily, in ten to twelve weeks. It is excellently adapted as a brief introduction to demonstrative geometry for classes which have not had an adequate amount of concrete geometry in the grammar grades.

The theoretical course consists of 66 theorems and 17 problems of construction, 83 propositions in all. One is pleasantly surprised to find touches of originality in the treatment of a subject so thoroughly explored as theoretical geometry for beginners. Though we may not approve the manner of treatment in all particulars, one is forced to recognize that the work is that of a careful student and wide-awake teacher of geometry. Some of the most noticeable characteristics of the theoretical course are:

1. Elegance of proofs combined with simplicity in notation. The principle of continuity is tacitly assumed and frequently applied in dealing with limiting cases. Thus the theorem that the tangent to a circle is perpendicular to the radius drawn to the point of contact (Proposition 44) is derived by considering the

limiting value of the exterior angle of the isosceles triangle formed by a secant and the radii to its points of intersection with the circumference. Again, the angle formed by a tangent and a chord through the point of contact is treated as the limit of the exterior angle of the triangle formed by the chord and the two sides of an inscribed angle. Whenever possible, triangles, angles and lines are referred to by single letters or marks, an improvement of inestimable value over the cumbrous notation commonly employed. For instance, in the proof of the fundamental theorem on transversals (Proposition 51) the following figure is employed



whence the equality of the various parts is expressed by such simple statements as:

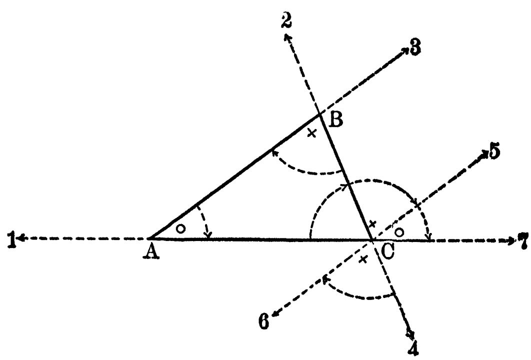
"All the ∠s marked / are equal; also all the ∠s marked // are equal; and the sides marked /// are equal; ∴ all the small △s are congruent, and so all their sides marked with a dot are equal; etc."

2. With two exceptions (Propositions 48 and 49) indirect proofs have been avoided. This is to be regretted, not only because in some cases (Euclid I, 25 ; I, 29 ; III, 21 ; III, 22.) the indirect proof excels all others in simplicity, but chiefly because of the intrinsic importance of the *reductio ad absurdum* principle with which the student should be made familiar as an instrument in exact reasoning.

3. Incommensurables are consistently ignored. Quantities of the same kind are treated as if they necessarily possessed some common measure. This procedure seems to us unwarrantable. Simplicity of treatment is purchased at too great a cost when it inculcates erroneous conceptions or does violence to truth itself. Better than to ignore intrinsic difficulties, is to

prepare for them by honorable recognition, though their treatment must of necessity be deferred to a later period if not indefinitely. It is easy to show even to beginners that the side of a square and its diagonal have no common measure, and thus to lay the foundation of a conception which permeates the whole of mathematics.

4. All axioms are introduced surreptitiously. One seeks in vain for explicitly assumed first principles. The angle sum of a triangle is proven equal to a straight angle by assuming that when the line lying along CA has been turned in succession about the points C, B and A until it again lies along CA the resultant turning is a half revolution. This of course is a fallacy.



Suppose that $C1$ has been turned about C clockwise until it lies along $C2$. Now while BC turns about B clockwise till it lies along BA, let $C4$ be turned about C through an equal angle and let its resulting position be along 56. The angles marked $\times$ are then equal. Let $A3$ now take its last turn about A until it lies along AC, and let $C5$ turn through an equal angle about C. We cannot say that $C5$ will take the direction of AC, without assuming the equality of the angles marked 0. This assumption justifies the conclusion that the sum of the turnings about C, B and A is equal to the straight angle $AC7$. But the assumption is no less than the theorem that two lines which make equal angles with one transversal make equal angles with every other transversal, the equivalent of Euclid I., 29.

The same theorem is smuggled into our author's definition of parallel lines. Parallel straight lines are defined as those which can, without turning, be made to lie in the same straight line. Straight lines are defined as lines of which two cannot enclose a space. The phrase "can, without turning, be made to lie" means, as we gather from proposition 23, that they can be moved at a constant angle along a transversal. Along what transversal? Some specific one? Evidently not, for then the definition would be useless. But if along any transversal, the assumption is once more the equivalent of Euclid I., 29.

Finally the scope of the theoretical course is far too limited to commend itself as suited for American requirements. Moreover the selection of the propositions has not always been judicious. Limited exclusively to plane geometry and further to that small portion of it treated by Euclid, the course ignores many of the most important propositions which we are accustomed to find treated in American books. This weakness may be justified from an English point of view, though our author has parted company with Euclid in every other particular. But we also search in vain for some of the most important propositions of Euclid, such as Euclid II, 12; II, 13; IV, 10–15. The omission of a theorem so fundamental as that which expresses the square on one side of triangle in terms of the other two sides and the projection of one of these sides upon the other, can hardly be justified on any grounds. On the other hand we fail to see the need for the proposition on medial section (proposition 70), when by the omission of Euclid IV, 10, we deprive the student of its application. Still less seems there be a sufficient reason for introducing Ptolemy's theorem (proposition 60), which is of little other than historic value. As between Euclid II, 12, 13, and Euclid VI, B, the omission of the latter would have purchased brevity at the lesser cost.

We have given a more detailed review of this book than an English elementary text-book may seem to deserve, for two reasons. First, because of the unusual interest that is manifested at present in anything that aims at improvement in geometrical teaching, and second, because this little book bears a clear and distinct message to American teachers, a message that will be welcomed by thousands of teachers of geometry who are looking for light in the direction to which this book points and who will find it a valuable aid in their work in the class-room.

ROBERT E. MORITZ.

THE UNIVERSITY OF NEBRASKA,
April 15, 1904.