

ON SOME TENDENCIES IN GEOMETRIC INVESTIGATIONS.*

REMARKS ADDRESSED TO MY STUDENTS.†

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I.

CHASLES at the end of his *Aperçu historique* says: "Dans la géométrie ancienne les vérités étaient isolées; de nouvelles étaient difficiles à imaginer, à créer; et ne devenait pas géomètre inventeur qui voulait.

"Aujourd'hui chacun peut se présenter, prendre une vérité quelconque connue, et la soumettre aux divers principes généraux de transformation; il en retirera d'autres vérités, différentes ou plus générales; et celles-ci seront susceptibles de pareilles opérations; de sorte qu'on pourra multiplier, presque à l'infini, le nombre des vérités nouvelles déduites de la première: toutes, il est vrai, ne mériteront pas de voir le jour, mais un certain nombre d'entre elles pourront offrir de l'intérêt et conduire même à quelque chose de très-général.

"Peut donc qui voudra, dans l'état actuel de la science, généraliser et créer en géométrie; le génie n'est plus indispensable pour ajouter une pierre à l'édifice."

A half-century has passed, and during this period geometry has made immense progress.‡ The study, projectively, of

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† Yielding to the kind persistence of the editor of the *Rivista di Matematica*, I repeat here, connectedly and with some additions, certain considerations which from time to time I have had occasion to present at the university to students specializing in mathematics, and which especially aim to caution those who wish to devote themselves to scientific research against certain faults and errors into which young investigators — and particularly young geometers — easily fall. Such remarks will not seem inappropriate at the present time, when many young men in Italy are engaged with the study of geometry. But by their nature and scope, they can present interest and novelty only to beginners: to them only may this paper be of some value.

‡ To get an approximate idea of the wealth of research which has been

algebraic curves and surfaces of the lowest orders was followed by the general theory of curves and surfaces of any order ; with the extension of the concept of geometric *elements* came that of *algebraic manifolds* or *spreads* and of the kinds of *space* in which they are considered. New types of problems have presented themselves. And the projective transformations to which principally Chasles alluded have found their generalization in the infinitely broader conception of the *algebraic transformations*.

Thus, owing to the exceeding growth of geometric research as well as of methods of transformation, grew also proportionately the facility mentioned by the great French geometer of increasing without limit the number of new propositions, of generalizing and creating in geometry. And this facility which, at least apparently, is greater in this science than in analysis, in mathematical physics, etc., induces many young men, especially in Italy, to give geometry the preference over other branches of mathematics, some of which indeed are very sadly neglected by our students.

But facility is a bad counsellor ; and often the work to which it leads the beginner, while it may serve as training, as preparation for original research, will not deserve to see the light. In the innumerable multitude of scientific publications geometric writings are not rare in which one would seek in vain for an idea at all novel, for a result which sooner or later might be of service, for anything in fact which might be destined to survive in the science ; and one finds instead treatises on trivial problems or investigations on special forms which have absolutely no use, no importance, which have their origin not in the science itself but purely in the caprice of the author ; or one finds applications of known methods which have already been made thousands of times ; or generalizations from known results which are so easily made that the knowledge of the latter suffices to give at once the former ; etc. Now such work is not merely useless ; it is actually harmful because it produces a real incumbrance in the science and an embarrassment for more serious investigators ; and because often it crowds out certain lines of thought which might well

made in this period the student need only glance at the historical monograph by Loria : "Il passato e il presente delle principali teorie geometriche," *Memorie della R. Accademia delle Scienze di Torino*, 2d series, vol. 38. [2d edition, Turin, 1896 ; German translation by Fritz Schütte, Leipzig, 1888.]

have deserved to be studied. Better, far better, that the student, instead of producing rapidly a long series of papers of such a nature, should work hard for a long time on the solution of a single problem, provided it is important: better one result fit to live than a thousand doomed to die at birth!*

II.

But when is a question *important*? When does it deserve to be made the object of study?

It is impossible to give a precise answer to this question. The importance of a result is largely relative, is judged differently by different men, and changes with the times and circumstances. It has often happened that great importance has been attached to a problem merely on account of the difficulties which it has presented; and indeed if for its solution it has been necessary to invent new methods, noteworthy artifices, etc., the science has gained more perhaps through these than through the final result. In general we may call important all investigations relating to things which in themselves are important; all those which have a large degree of generality, or which unite under a single point of view subjects apparently distinct, simplifying or elucidating them; all those which lead to results that promise to be the source of numerous consequences; etc.

The study of the great masters is perhaps the best thing to recommend to the student who wishes to prepare himself to judge of the importance of problems. For it is precisely in the choice of these that the great minds have always shown themselves masters; and even when they have taken up very special problems, they have shown in what way those could become important. And here, in corroboration of the preceding, I will quote the words of Beltrami: † “Students should learn to study at an early stage the great works of the great masters instead

* To students who are looking toward the doctorate in mathematics it is well to say frankly that science should not be thought of as a profession in which all can succeed; though it be true that genius is no longer essential to produce useful results, still a certain aptitude is necessary; and he who knows himself to be without it should, with that veneration and sacrifice which science demands, renounce scientific research. Why should a young man, who could perhaps teach successfully the elementary mathematics and study thoroughly the numerous and important pedagogical questions which present themselves in his teaching, neglect such studies, in order to take up researches in higher mathematics which are not adapted to his type of mind?

† *Giornale di matematiche*, vol. 11, p. 153.

of making their minds sterile through the everlasting exercises of college, which are of no use whatever, except to produce a new Arcadia where indolence is veiled under the form of useless activity. . . . Hard study on the great models has ever brought out the strong; and of such must be our new scientific generation if it is to be worthy of the era to which it is born and of the struggles to which it is destined."

In such studies one should ever keep before him this other object: *to broaden as much as possible his own knowledge*. He who is interested only in works relating to the limited field which he is studying, will in the end give undue weight to questions which do not seem so important to another, who with broader knowledge looks at the subject from a higher point of view. It should be the aim of the student by an extended study of the best works in all branches, to attain a wider horizon with regard to the whole science.

III.

In a letter of Jacobi to Legendre (July 2, 1830) concerning a report by Poisson upon the *Fundamenta Nova*, we find:

" . . . M. Poisson n'aurait pas dû reproduire dans son rapport une phrase peu adroite de feu M. Fourier, où ce dernier nous fait des reproches, à Abel et à moi, de ne pas nous être occupés de préférence du mouvement de la chaleur. Il est vrai que M. Fourier avait l'opinion que le but principal des mathématiques était l'utilité publique et l'explication des phénomènes naturels; mais un philosophe comme lui aurait dû savoir que le but unique de la science, c'est l'honneur de l'esprit humain, et que sous ce titre, une question de nombres vaut autant qu'une question du système du monde. . . ."

There should be no doubt in regard to this view of Jacobi: *Science should be given absolutely the greatest liberty*; and in particular, we can not require it always to keep in sight the practical applications.

On the other hand, in the existing state of mathematical research in Italy, now that among our young mathematicians only very few are engaged in the study of mechanics, mathematical physics, etc., it is proper to recall the following very true words, which Fourier himself wrote eight years before in the preface of the *Théorie analytique de la chaleur*:

"L'étude approfondie de la nature est la source la plus féconde des découvertes mathématiques. Non seulement cette

étude, en offrant aux recherches un but déterminé, a l'avantage d'exclure les questions vagues et les calculs sans issue : elle est encore un moyen assuré de former l'analyse elle-même, et d'en découvrir les éléments qu'il nous importe le plus de connaître, et que cette science doit toujours conserver : ces éléments fondamentaux sont ceux qui se reproduisent dans tous les effets naturels."

Is it necessary to give examples in confirmation of the above ; to record how the greater part of mathematical problems, from the simplest to the most advanced, have had their origin in applications to nature ; how the theories of ordinary and partial differential equations, the trigonometric series, various new functions which later became of great importance in analysis, all originated in or received continual stimulation from mathematical physics or celestial mechanics ; how the theory of the potential (and hence the subject of electrodynamics, etc.) has been united in a remarkable way with the theory of functions of a complex variable, and in particular of algebraic functions and their integrals, chiefly through the work of Riemann (that worthy successor of Gauss and Dirichlet) and his followers, so that one of these, Klein,* was able to refer to certain electrical phenomena to show the existence of the various abelian integrals ? Or shall we recall how often the study of conics and quadrics has developed from that of natural phenomena; or how the null-system and the linear complex and congruence of straight lines, etc., have presented themselves in the mechanics of rigid bodies, while mechanics in turn has received extraordinary help from the geometry of straight lines (thanks chiefly to Plücker, Klein and Ball†). To give even a faint idea of the many different ways in which the problems of nature have spurred on the mathematical sciences to further progress, would be impossible ; equally impossible would it be to enumerate all those great men, some of whom are still living and laboring, who have known how to combine the most advanced researches in pure mathematics with their applications to physics, astronomy, engineering.

Students who wish to devote themselves to scientific re-

* Ueber Riemanns Theorie der algebraischen Functionen und ihrer Integrale (Leipzig, 1882).

† The Theory of Screws (Dublin, 1876) [and A Treatise on the Theory of Screws (Cambridge, 1900)]. See also the work of Gravelius : Theoretische Mechanik starrer Systeme auf Grund der Methoden und Arbeiten Balls (Berlin, 1889).

search should keep before them these facts and these examples, and should study the pure theories at the same time with their applications. Among the many advantages which they will derive therefrom will always be this, that when they are without any other means of judging of the importance of a certain theoretical problem, there will ever remain the one furnished by its possible applications (which, as I have already said, though not *necessary*, is certainly *sufficient* to justify a scientific investigation).

IV.

What has been said of the relation between pure and applied mathematics applies to a far greater degree to the two principal subdivisions of pure mathematics, analysis and geometry. The method of coördinates serves to pass from one to the other and unites them intimately, or rather so welds them together that we may say that every advance in the one means an advance in the other. The whole of differential geometry stands in proof of this; in particular the manifold relations which it establishes between differential equations and curves, surfaces, the connexes introduced by Clebsch,* etc.; and the theory of continuous groups of transformations recently created by Lie has not only rendered great service to the theory of differential equations, but has on the other hand solved noteworthy geometrical problems, throwing light, for example, on the foundations of geometry.† The geometry of *algebraic* manifolds coincides, as is shown by the very name and definition of these forms,‡ with the analysis of algebraic functions and the related transcendental functions: thus projective geometry is the equivalent of the modern algebra of linear transformations (of invariants), etc.

* Concerning the researches of this great geometer on the connexes his biographers say (*Math. Annalen*, vol. 7, p. 50). "Er hat mit ihnen dem Grundzuge seiner mathematischen Denkweise noch einmal Ausdruck gegeben, welche die *Mathematik nicht als eine Reihe geschiedener, einander fremder Disciplinen, sondern als einen lebendigen Organismus erfassen wollte.*"

† Cf. Poincaré, "Sur les hypothèses fondamentales de la géométrie," *Bull. Soc. Math. de France*, vol. 15 (1887); and especially Lie, "Ueber die Grundlagen der Geometrie," *Berichte der k. sächs. Ges. d. W.*, Leipzig, 1890. [Lie, *Theorie der Transformationsgruppen*, III. Abschnitt 5, 1893.]

‡ A manifold or spread is said to be *algebraic* when its elements are the totality of those whose coördinates satisfy given algebraic equations (in which also indeterminate parameters may appear rationally). With this definition we may also include the notion of algebraic correspondence. [Cf. my *Introduzione alla geometria sopra un ente algebrico semplicemente infinito*, *Annali di Matematica* 2nd ser., vol. 22 (1893).]

And the theory of algebraic equations and of groups of substitutions throws light on many important problems of geometry, and in turn derives from the latter many useful representations and suggestions.* Etc.†

This great variety of bonds between analysis and geometry and the resulting necessity of studying them both together, and of not confining oneself to one alone of these two directions, the analytic or the synthetic, has always been recognized by the great mathematicians. Both Lagrange and Monge, for example, emphasized this in their courses in analysis and geometry at the Ecole normale (1795); and the latter in his *Géométrie descriptive* wrote the following, which evidently applies to the whole of geometry :

“ . . . Ce n’ est pas sans objet que nous comparons ici la géométrie descriptive et l’algèbre ; ces deux sciences ont les rapports les plus intimes. Il n’y a aucune construction de géométrie descriptive qui ne puisse être traduite en analyse ; et lorsque les questions ne comportent pas plus de trois inconnues,‡ chaque opération analytique peut être regardée comme l’écriture d’un spectacle en géométrie.

“ *Il seroit à désirer que ces deux sciences fussent cultivées ensemble ; la géométrie descriptive porterait dans les opérations analytiques les plus compliquées, l’évidence qui est son caractère ; et, à son tour, l’analyse porterait dans la géométrie la généralité qui lui est propre.*”

And in more recent times Clebsch wrote regarding these two branches :§

“ *Beide zusammen umfassen erst in Verein und Ergänzung das Ganze mathematischer Forschung, und es vermag keine von beiden auf die Dauer ohne schwere Schädigung ihres eigenen Wesens die Begleitung und den Einfluss der andern zu entbehren.*”

* Cf. for example the chapter on geometrical applications in the *Traité des substitutions et des équations algébriques* of C. Jordan.

† We recall also Klein’s *Programm : Vergleichende Betrachtungen über neuere geometrische Forschungen* (Erlangen, 1872) [reprinted in *Math. Annalen*, vol. 43 ; translated into Italian by G. Fano, *Annali di Matematica*, 2d series, vol. 17, and into English by M. W. Haskell, *Bulletin of the New York Math. Society*, vol. 2 (1892), p. 215].

‡ This restriction may be removed by the use of spreads of n dimensions. I shall return to this later.

§ “ *Zum Gedächtniss an Julius Plücker,*” *Abhandl. der k. Ges. d. W. zu Göttingen*, vol. 16 (1871) [reprinted in Plücker, *Gesammelte mathematische Abhandlungen*, Leipzig, 1895] translated into Italian by Professor Beltrami, *Giornale di Matematiche*, vol. 11. Compare this extract with that from the biography of Clebsch cited in a preceding note.

The same ideas are being advanced in lectures and by example by one of the greatest masters that Germany now boasts, Klein; indeed the whole school of which he is the head proves the advantage of possessing at once a knowledge of both analytic and geometric methods.

In Italy, however, the conditions are only too different.* The division of pure mathematics into analysis and geometry is made by young mathematicians in such a precise way† that it can be said of only a few that they are studying both. We have young analysts and young geometers; but young men who consider the whole of mathematics as their proper field, following the great examples that we have even in Italy, are very few. For this reason when I turn to my students of geometry, I feel it my duty to urge them very warmly to take up the study of analysis. A student who wishes nowadays to study geometry by dividing it sharply from analysis, without taking account of the progress which the latter has made and is making, that student, I say, no matter how great his genius, will never be a whole geometer.‡ He will not possess those powerful instruments of research which modern analysis puts into the hand of modern geometry. He will remain ignorant of many geometrical results which are to be found, perhaps implicitly, in the writings of the analysts. And not only will he be unable to use them in his own researches, but he will probably toil to discover them himself, and, as happens very often, he will publish them as new, when really he has only rediscovered them.

After the examples already given it seems useless to add more to justify these views. But there is one which I wish to cite, because it is instructive in the highest degree: *the geometry on an algebraic curve*.§ The idea of this geometry, *i. e.*, the properties of an algebraic curve which remain invariant under a rational transformation of the curve into itself, is first found

*[These words, as also the following, true in 1891, must now be modified. The present conditions in Italy are much better.]

† We must confess that they find the division already thus made in the catalogue of courses offered at the university. They find placed in opposition *projective geometry* and *analytic geometry*! *higher analysis* and *higher geometry*!

‡ Remember that "geometer" in the wider sense of the term is synonymous with "mathematician."

§ [Cf. my *Introduzione* above referred to, and especially the very important report of Brill and Noether, "Die Entwicklung der Theorie der algebraischen Functionen in älterer und neuerer Zeit," *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 3 (1893).]

in a work of analysis, in the great memoir of Riemann on the theory of abelian functions. It is here that we first find fully developed the notion of *genus* (it had already been introduced in substance by Abel in his capital memoir on his transcendents) and the demonstration that it is unchanged by such transformation. It is here that we find that representation of an algebraic function as the sum of abelian integrals of the second kind, which (thanks to Roch who completed the calculation of Riemann) gave to modern geometry one of its most important and fruitful theorems. It is here finally that we find so many noteworthy theorems, some of which when clothed in geometric form may well appear to be new to young geometers who have not studied thoroughly this profound and immortal work. New results in the geometry on a curve may also be found in the lectures of Weierstrass on abelian functions. And all through the geometry of algebraic curves the analytic researches on algebraic functions and their integrals have had applications of the greatest importance. It is sufficient to recall in this connection Abel's theorem (the natural bridge, as it has been called, between the algebraic and transcendental sides of this theory) and the inversion problem with which it is identified and the geometric applications of both of which Clebsch has exhibited.

Nor is the influence of Riemann on the applications of analysis to geometry confined to this. Indeed to his influence may be traced in many ways various new classes of transcendental functions, which are now defined and studied apart from the abelian functions. So Poincaré introduced certain uniform, analytic functions of one variable, which he called *fuchsian functions*, by means of which he is able to express the coördinates of the points of any algebraic curve. And this representation has already been used in studying curves;* it would however be desirable to see if more use can not be made of it. These functions are special cases of the more general class of *linearly automorphic functions*, as Klein calls them, *i. e.* of functions of one variable which remain unchanged when the variable is transformed by an infinite discontinuous group of linear transformations.† It is well worth while for young geometers to

* Cf. *e. g.* Humbert, "Application de la théorie des fonctions fuchsianes à l'étude des courbes algébriques," *Journ. de mathém.*, 1886.

† Cf., besides the fundamental articles of Poincaré (in the first volumes of the *Acta Mathematica*), "Neue Beiträge zur Riemann'schen Functionentheorie," by Klein (*Math. Annalen*, vol. 21) [also Fricke-Klein, *Vorlesungen über die Theorie der automorphen Functionen*, Leipzig, 1897; Forsyth, *Theory of Functions of a complex Variable*, 2nd ed., Cambridge, 1900, last chapter.]

penetrate into the vast territory which these and other functions, both of one and of more variables, which are being introduced daily, open up for study; for it is certain that, even if they wish to confine their research to the field of geometry, the analytic knowledge thus acquired will be of the greatest service, and may perhaps give the key for the solution of most difficult geometrical problems.

V.

Here we may with propriety say a few words concerning the use of the *purely synthetic method* in geometrical research.

In any progressive development, either of a living organism, or of a social order, or of a science, two tendencies among others are always apparent: one to make ever finer distinctions in the structure and functions of the various parts, the other toward an ever greater connection and dependence among them.* It follows that the growth of the sciences, while on the one hand it increases their number by forming so many new sciences out of theories which before were parts of a single science, on the other hand continually increases the bonds between them and the means of mutual assistance which they can render.†

It is thus that among the many branches of mathematical progress in this century, aside from the new and instructive relations and applications established among them and in particular between analysis and geometry, was signalized the purification so to speak of these sciences, that is to say, the construction of an analysis based solely on the notion of number without admitting geometrical (or mechanical) considerations, and the construction of a projective geometry which makes no use of coördinates and avoids entirely the notion of number in the properties of position.

These *pure* methods are indeed of the greatest importance. It is beyond doubt that the mathematician can not be fully

* Or, as Herbert Spencer puts it in his general formula of evolution (First principles, 6th edition, p. 367), "a passage from an indefinite, incoherent homogeneity to a definite, coherent heterogeneity."

† Hence results what is perhaps the greatest difficulty which the student of today must meet, that of reconciling the division of labor, which is inevitable on account of the great development of the sciences, with the necessity of keeping up with the progress in several of them which are all intimately connected. Unfortunately their mental capacity and their physical strength make it possible for only a few to do this in an entirely satisfactory way. It is the young man who should use all his powers to approximate as near as possible to the scientific culture of which I speak; the studies pursued in youth are the ones which leave the deepest impression on the mind!

satisfied with the knowledge of a truth, unless he has succeeded in deriving it with the greatest simplicity and naturalness from the smallest possible number of known theorems, of independent postulates, avoiding every hypothesis and every means of demonstration which is not absolutely required for the purpose. By so doing we often gain besides the scientific advantage also a pedagogic advantage, in so far as a smaller amount of preliminary knowledge will be required of the student. Further we may note as a general principle that, when in our researches we adopt a single instrument, to the exclusion of every other, it will often happen that we make it more refined and more perfect, thus rendering it ever better adapted to new problems. And in this way the synthetic method, which was brought to perfection in the great works in which modern geometry was built up, has gained wonderful triumphs that show clearly the advantages which often result from its systematic use.

But in this line of thought the young geometer should be careful not to exaggerate. The heroic age, so to speak, of synthetic geometry, in which the object was not merely to give new results to the science, but in which all, from Poncelet to Steiner, from Chasles to Staudt, were engaged in the struggle to prove the usefulness of the geometric method to the analysts who were not willing to recognize it,—that age has passed;* and nowadays battle is no longer necessary. That some one should be engaged in establishing synthetically results which others have already obtained analytically, is often very useful for reasons just hinted at; but in doing this he should choose problems which present particular importance and noteworthy difficulties; and then, these conditions being equal, he should always give the preference to investigations which are entirely new, and not become a mere translator of analytic writings into synthetic. And in his original research he should not feel obliged to cling exclusively to the geometric method, since ultimately the important things in the science are the results (unless the question is one of inventing new methods), and it

* And now seems to be the proper time to write its history!—By the study of the great geometers some scholar should become inspired to search and relate how and by whose work have been obtained the most important advances in modern geometry. A detailed history of these advances would constitute a work of the highest interest. [This history, in part at least, has now been written. Cf. Ernst Kötter, “Die Entwicklung der synthetischen Geometrie von Monge bis auf Staudt,” *Jahresber. d. Deutschen Math.-Vereinigung*, vol. 5 (1901).]

would be folly to avoid certain geometric questions merely because in the present state of the science they cannot be treated without the use of analytic instruments. Artificially to create difficulties, even though it may serve to sharpen the intellect and may therefore be of some use, is nevertheless not a plan to be followed in general by one whose chief object is the progress of the science.* For every piece of research one should choose freely the method which seems best adapted; often it will be found convenient to alternate between the synthetic method which appears more penetrating, more enlightening, and the analytic which in many cases is more powerful, more general or more rigorous; and it will frequently happen that the same subject will not be quite clear from all points of view unless treated by both methods. † Certainly here also the individual tendencies will make themselves felt and the same problem will seem to one more adapted to the synthetic treatment, to another to the analytic; thus Steiner and Cremona arrived at certain results synthetically at the same time that Sylvester and Clebsch obtained them analytically. But the most important thing is that the geometer should not be the slave of a single method and that he should put himself in a position where he can make use of any instrument to obtain an important result.

VI.

Just as purity of method takes a secondary place, when we are engaged merely with the discovery of a new result, so it often happens that in a first investigation we must sacrifice rigor (a sacrifice much more serious, since we are dealing with mathematics!). ‡ Many times a scientific truth is

* For example, after Staudt has given a complete geometric theory of the imaginary elements, proving that all the fundamental properties of projective geometry hold for them, there is no reason in scientific research in this field (I am not speaking of those which are essentially pedagogical) for excluding imaginary elements, and still less for introducing them only in *couples* (of conjugate imaginary elements). It is understood that here, as always, I speak in most general terms, admitting the necessary exceptions.

† It is for this reason that in the present state of the science those treatises (or those courses) in which a subject is unfolded with a variety of methods, seem to me *in general* to be preferred to those in which a single method is adopted; thus algebraic curves, surfaces, etc., should, it seems to me, be presented to students analytically as well as synthetically, generally alternating the two methods according to the questions.

‡ We should not confuse lack of rigor in the methods with errors in reasoning or in the results. Errors are encountered only too often nowadays in

placed as it were on a lofty peak, and to reach it we have at our disposal at first only hard paths along perilous slopes whence it is easy to fall into abysses where dwells error; only after we have reached the peak by these paths is it possible to lay out safe roads which lead there without peril. Thus it has frequently happened that the first way of obtaining a result has not been quite satisfactory, and that only *afterwards* did the science succeed in completing the demonstration. Certainly also a mathematician can not be really content with a result which he has obtained by non-rigorous methods; he will not feel sure of it until he has rigorously proved it. But he will not reject summarily these imperfect methods in the case of difficult problems when he is unable to substitute better ones, since the history of the science precisely shows what service such methods have always rendered.

For example, the introduction of the imaginary and the infinitely distant elements into geometry proved to be most useful before it was justified with perfect logic. And Poncelet, who learned to employ to great advantage the principle of continuity for studying geometric figures, never succeeded in establishing it rigorously in spite of the many pages which he wrote on the subject.* Similarly in various, chiefly synthetic researches on curves and surfaces the writers have profited much by the considerations of infinitely near, or consecutive, points, of tangents, of r -ple points as the intersection of r branches, etc., which were far from being rigorous, at least in general and in the way in which they were presented. The same may be said of many processes of *enumerative geometry* † (in which we may

scientific work and are due generally to the haste with which the latter is produced. It is therefore well to advise young men not to show excessive haste in publishing their results! But the question of rigor, of which we speak above, is quite different.

* Cf. especially: "Considérations philosophiques et techniques sur le principe de continuité dans les lois géométriques" in vol. 2 of his *Applications d'Analyse et de Géométrie*; and the *Traité des Propriétés projectives des Figures*. From the introduction to this treatise I quote the following bit which applies to what we were saying above:

"N'est il pas, pour le moins, aussi nécessaire d'enseigner les ressources employées, à diverses époques, par les hommes de génie, pour parvenir à la vérité, que les efforts pénibles qu'ils ont été ensuite obligés de faire pour les démontrer selon le goût des esprits ou timides ou peu capables de se mettre à leur portée?"

"Enfin, quel mal pourrait-il en résulter, surtout si l'on se montrait sévère à conclure, si l'on ne se payait jamais de demi-aperçus, si l'on n'admettait jamais l'*analogie et l'induction*, qui sont souvent trompeuses . . . ?"

† Cf. the very important work by Schubert: *Kalkül der abzählenden Geometrie* (Leipzig, 1879), in which these processes are collected and employed systematically.

include the last examples cited), such as the principle of counting constants and especially that of the invariance of number, in the application of which it is often assumed without a complete proof, that the number sought depends only on certain others, and therefore does not change (or become infinite) when they do not change; methods which have led able scientists, such as Jonquières,* Schubert and others, to splendid results, among which are many that the science would still be at a loss to reach by a more rigorous route.† Finally I will record that sometimes recourse has been had even to drawings and models of geometric forms in order to *see* certain properties (especially of form or of reality) which no one knew how to obtain by deductive reasoning alone.‡

Now in making use of similar means of investigation, the young investigator should be careful to remember that here, as in the difficult ascent of perilous heights, in order to avoid falling into an abyss or into error, he needs skill, prudence and practice. And having availed himself with the utmost caution of those means for his discoveries, he should seek to substitute for them rigorous proofs. But, as I have already said, I do not believe that he should refrain from engaging upon a line of research or from publishing a result, merely because he has been unable to proceed with absolute rigor. The greatest caution, I repeat, the largest number of verifications, etc., should be employed in order to avoid error; and also here the example of able mathematicians should help him to determine when he may have faith in a result which he has obtained; but then he can make

* Cf., for example, the determination of the number of groups of a linear series of point groups of a curve which have points of a given multiplicity; a determination contained substantially in the "Mémoire sur les contacts multiples des courbes de degré r , qui satisfont à des conditions données, avec une courbe fixe, etc.," by de Jonquières, *Crelle's Journal*, vol. 66 (1866).

† Speaking to young men, I like to cite among the more important results thus obtained, the one recently published by a young man, viz., the number of linear series of given index on a curve of given deficiency (cf. two notes by Castelnuovo in the *Rendiconti della R. Acc. dei Lincei*, summer, 1889). [Cf. the analytic justification of the geometric enumerative process of Castelnuovo in Klein, *Riemann'sche Flächen II.*, p. 110 ff., lithogr. lectures, Göttingen 1892.]

‡ And there are other cases in which results relating to mathematics have been obtained by the use of physics. Even this is permissible; but it should be understood that the result thus obtained, though it be *scientific (physical)*, is not therefore *mathematical*, will have only a relative and approximate value; however, it will prepare the way for the mathematical result, *i. e.*, for one established by a complete mathematical proof.

it known. Only it is necessary to unite *honesty* with caution, *i. e.*, to call particular attention to where the method used is open to question, so that no one may be induced to believe blindly in the result, but that it may be in itself an invitation to seek a more complete proof.*

VII.

At the outset of these remarks, following a quotation from Chasles, we spoke briefly concerning the extent and the fundamental importance which the notions of correspondence and algebraic transformations have in the geometry of algebraic spreads, and also of the facility with which by means of these we may obtain new forms and new propositions.

As regards their importance, we may consider this more particularly from three different points of view, corresponding to the following three functions of the correspondences themselves: 1° that of characterizing certain branches of geometry; † thus we have a projective geometry, in which the fundamental group of transformations is that consisting of the projectivities, a geometry of birational transformations, *i. e.*, the geometry of birational transformations in the plane or in space, etc., the geometry *on a curve* (which with the aid of function-theoretic considerations is made to correspond to the geometry of the conformal transformations of surfaces), the geometry *on a surface*, etc.; 2° that of transforming forms whose properties are known into others, of which we shall thus learn new properties; 3° that of generating new spreads by making use of algebraic correspondences among given forms, such as the loci of self-corresponding elements, or else the loci of the intersections (or the connections) of homologous elements, etc.

The second of the above divisions is suggested by the first, when in studying the properties of some geometric object from

* Among the things which must often be neglected in the first discovery of new theorems should be mentioned that of the exceptional cases. We frequently say that a proposition is true *in general*, without further specifying what is meant thereby, what are the *exceptions*. Of course this again lacks that exactness which is the greatest pride of mathematics. In the geometry of algebraic spreads a sufficiently precise meaning may be given to the phrase *in general*; we there mean that the exception takes place only in the case of those spreads whose coördinates (or parameters on which they depend algebraically) satisfy certain non-identical algebraic relations. If it is possible actually to exhibit these relations, that is, the conditions which characterize the exceptional cases, the mathematical ideal will be completely attained. But often we must content ourselves with merely the generic statement.

† Cf. Klein's Programm, already cited.

a certain point of view, *i. e.*, the properties which remain unchanged for a certain class of geometric transformations, we apply the latter to simplify the object of study, to make its study easier. This is the case, for instance, in the method of central projection as used in projective geometry; or when, in the study of the geometry on a curve, we make use of certain *normal* curves; or when the properties of certain linear systems of curves in a plane (in the geometry of the birational transformations of the plane) are derived from those of linear systems of lowest order. If, however, we compare the *Traité des Propriétés projectives des Figures* of Poncelet with the later treatises, for example with the *Geometrie der Lage* of Staudt,* we shall note how the importance of this method of demonstration has grown less in projective geometry, and how it has been deemed better to study directly the more general case, rather than to study first a particular and thence derive the general by projective transformations. And something analogous may already be seen in certain questions of the geometry of birational transformations.

This does not mean at all that the transformations lose in any way their importance. Far from it! It is merely the mode of application which changes. It is true, for example, that the study of *particular* rational curves and surfaces by means of their parametric representation, or their representation on a straight line or plane, and likewise the study of *particular* birational correspondences between two planes or two spaces, no longer commands the interest it had some twenty years ago. At that time they led to the consideration of new objects and new and very interesting problems, and the best geometers turned to them with profit. Nowadays to multiply the applications of the methods which were then invented is in general of secondary importance, unless the question is one of certain exceptional cases which present new difficulties. Given the linear system of plane curves representative of a surface, or given the homaloidal system of curves or surfaces which define a birational transformation of the plane or of space, it is in many cases nothing but a class-room exercise to derive therefrom the properties of the surface or of the birational transformation. And so in general, just as it is easy to imagine new loci and new geometric transformations, so also is it easy

* Translated into Italian by Pieri (Turin, Bocca, 1889).

to invent applications of correspondences to obtain new truths. We take a point A , join it to B , take the polar with regard to C , let it intersect with D , take the homologous point with regard to E , etc., and finally from A we obtain a point (or other element) A' ; to the elements A of the first set will thus correspond the elements A' of a new set, and if A or A' is made to move in a certain way, A' or A will also move, etc. In this way a given figure or given property will be transformed into another, which will give rise to a third, and so on; and all without the least difficulty, mechanically as it were, with the regularity with which a pendulum swings.*

Now, though we remain in the field of algebraic correspondence, it is not with this sort of research that we should be engaged nowadays. There are indeed other problems of much greater importance!† The theory of the birational transformations of space is waiting for some one to attack certain general problems of prime importance, whose analogues for the plane have already been solved. It is waiting to be applied to the serious problem of the resolution of the higher singularities of twisted curves and surfaces, to the study of linear systems of surfaces, etc. The theory of the multiple correspondences between two planes or two spaces is still to be worked out. The study of algebraic correspondence between curves (distinct or coincident) is far from complete. And of that of the correspondences between two surfaces it may be said (in spite of a recent work by an able French mathematician) that it is still to be begun. In all of these fields may be found most vital questions to which the young geometer should address himself rather than to the exercises previously mentioned.

Even in the application of algebraic correspondences to the generation of geometric loci, important results have been found, and may still be found. But these correspondences must be chosen so as to lead to loci which interest by reason either of

* Hence one of my teachers (from whom I have received much of this advice which I am now giving to my students) used jokingly to call this kind of research *tic-tac-geometry*.

† [Several of the important questions which I have mentioned have since been discussed, in full or in part; the higher singularities of twisted curves and surfaces chiefly by myself and by B. Levi; algebraic correspondences between curves by A. Hurwitz; correspondences between surfaces by Picard, Painlevé, Castelnuovo-Enriques, Severi. Upon the geometry on an algebraic surface the not yet complete treatise of Picard and Simart, *Théorie des Fonctions algébriques de deux Variables indépendantes* (Paris, 1897, 1900, 1904) may be consulted with profit.]

their generality or of special circumstances. We may observe moreover that often the determination of the various characteristics (order, class, multiple points, etc.) of the forms thus defined does not present the least novelty or difficulty, because they are obtained by well known enumerative processes. The investigations therefore should not be limited merely to such a determination, but should include the examination of all the peculiarities of the loci obtained, with a view to solving the inverse question, *i. e.* of determining all the forms which may be generated in this way.

VIII.

Aside from the extension given to modern geometry by the use of transformations, we have already mentioned that derived from the enlargement of its field by the consideration of ever more extensive classes of spreads or systems. With this we may connect the consideration of the geometry of hyperspace, thanks to which the boundaries of geometry have been widened indefinitely. On this subject we may well say a few words,* on the one hand because during the past decade, especially after the appearance of the now famous memoir of Veronese,† several Italian geometers ‡ have turned to it with much energy, and chiefly the younger ones; on the other hand because there are still, in spite of this, some even in Italy who are unable to accord to geometry of n dimensions its proper place. §

* [In my article "Mehrdimensionale Räume," which is to appear in the Encyclopädie der mathematischen Wissenschaften will be found more historical and bibliographical notes. Since the appearance of the present paper multi-dimensional geometry has spread more and more, so that now (among mathematicians!) its opponents have become rare, who at one time were so common. Not only the geometers, but also the abler analysts, no longer hesitate to make use of hyperspace in their researches.]

† "Behandlung der proj. Verhältnisse" etc., *Math. Annalen*, vol. 19, 1882.

‡ We could count at least twenty. [Now many more.]

§ To complete what I shall say in the following, and to support it with the voice of authority, I will repeat here the 4th Note (Über Mannigfaltigkeiten von beliebig vielen Dimensionen) of Klein's Programm, already referred to:

"Dass der Raum, als Ort für Punkte aufgefasst, nur drei Dimensionen hat, braucht vom mathematischen Standpunkte aus nicht discutirt zu werden; ebensowenig kann man aber vom mathematischen Standpunkte aus jemanden hindern, zu behaupten, der Raum habe eigentlich vier, oder unbeschränkt viele Dimensionen, wir seien aber nur im Stande drei wahrzunehmen. Die Theorie der mehrfach ausgedehnten Mannigfaltigkeiten, wie sie je länger je mehr in den Vordergrund neuerer mathematischer Forschung tritt, ist, ihrem Wesen nach, von einer solchen Behauptung vollkommen

We may distinguish three ways in which hyperspace has presented itself to geometers; and to these correspond as many ways of defining the points of a linear space of n dimensions. In the first place, on the basis of the method of coördinates, we regard the points of a line, of a plane, or of space, as represented by analytic spreads composed of all possible values of one number, of two numbers, or of three, so that systems of equations in one, two, or three variables are represented by certain sets of points, etc. We are led naturally to extend the language of geometry to the case of any number n of variables, still using the word *point* to designate any system of values of n variables (the *coördinates* of the point), the word *space* (of n dimensions) to designate the totality of all of these points or systems of values, *curve* or *surface* to designate the spread composed of points whose coördinates are given functions (with the proper restrictions) of one or two parameters (*straight line* or *plane*, when they are linear fractional functions with the same denominator), etc. Such an extension has come to be a necessity in a large number of investigations,* in order as well to give them

unabhängig. Es hat sich in ihr aber eine Redeweise eingebürgert, die allerdings dieser Vorstellung entflohen ist. Man spricht, statt von den Individuen einer Mannigfaltigkeit, von den Punkten eines höheren Raumes, etc. An und für sich hat diese Redeweise manches Gute, insofern sie durch Erinnern an die geometrischen Anschauungen das Verständniss erleichtert. Sie hat aber die nachtheilige Folge gehabt, dass in ausgedehnten Kreisen die Untersuchungen über Mannigfaltigkeiten mit beliebig vielen Dimensionen als solidarisch erachtet werden mit der erwähnten Vorstellung von der Beschaffenheit des Raumes. Nichts ist grundloser als diese Auffassung. Die betr. mathematischen Untersuchungen würden allerdings sofort geometrische Verwendung finden, wenn die Vorstellung richtig wäre, — aber ihr Wert und ihre Absicht ruht, gänzlich unabhängig von dieser Vorstellung, in ihrem eigenen mathematischen Inhalte.

Etwas ganz anders ist es, wenn Plücker gelehrt hat, den wirklichen Raum als eine Mannigfaltigkeit von beliebig vielen Dimensionen aufzufassen, indem man als Element des Raumes ein von beliebig vielen Parametern abhängendes Gebilde (Curve, Fläche, etc.) einführt.

Die Vorstellungsweise, welche das Element der beliebig ausgedehnten Mannigfaltigkeit als ein Analogon zum Punkte des Raumes betrachtet, ist wohl zuerst von Grassmann in seiner Ausdehnungslehre (1844) entwickelt worden. Bei ihm ist der Gedanke völlig frei von der erwähnten Vorstellung von der Natur des Raumes; letztere geht auf gelegentliche Bemerkungen von Gauss zurück und wurde durch Riemann's Untersuchungen über mehrfach ausgedehnte Mannigfaltigkeiten, in welche sie mit eingeflochten ist, in weiteren Kreisen bekannt.

Beide Auffassungsweisen — die Grassmann'sche wie die Plücker'sche — haben ihre eigenthümlichen Vorzüge; man verwendet sie beide, zwischen ihnen abwechselnd, mit Vortheil."

* I will cite as examples only the investigations concerning the systems of algebraic equations (algebraic spreads in any space) of Salmon (Geometry of

the greatest generality, as to preserve in them the intuitive character of geometry. But it has been noted that in such use of geometric language we are no longer constructing truly a geometry, for the forms that we have been considering are essentially analytic; and that, for example, the general projective geometry constructed in this way is in substance nothing more than the algebra of linear transformations.*

A geometric method of arriving at the notion of hyperspace, following the ideas of Plücker, is to consider as elements (*points*) of a spread (of a *space*) geometric forms of ordinary space, such as groups of points, curves, surfaces, . . . which depend on any arbitrary number of parameters. It is thus that the straight lines of ordinary space may be thought of, according to Plücker, as *points* in a space of four dimensions. However, if in this representation it be desired to avoid exceptional elements, *i. e.*, if it be desired to represent *linearly* a manifold of ∞^n ordinary geometric forms by the points of a (linear) hyperspace, it is necessary that the manifold be linear. Thus an involution of points on a straight line of the n th species and any order (for example that composed of all the groups of n points), a linear system of ∞^n plane curves, or surfaces, or connexes, etc., is from this point of view a (linear) space of n dimensions, such that the spreads therein contained are nothing but partial spreads of the involution, or of the linear system in question. This means of representation offered itself spontaneously to the geometers who wished to go more deeply into the questions concerning infinite systems of plane curves (or surfaces).†

Three Dimensions and Modern Higher Algebra), the work of Noether, "Zur Theorie des eindeutigen Entsprechens algebraischer Gebilde von beliebig vielen Dimensionen," *Math. Annalen*, vol. 2 (1869), and the memoir of Clebsch, "Ueber eine Fundamentalaufgabe der Invariantentheorie," *Gött. Abhandlungen*, vol. 17 (1872)), referring for other instances to the historical monograph of Loria.

* This is a distinction, not a reproach. It is *mathematics* that is being made.

† Thus Cayley makes use of it in the first paragraph of his memoir "On the curves which satisfy given conditions," *Phil. Transactions*, vol. 158, 1867 [Coll. Math. Papers, vol. VI, p. 191], and he returns to the subject in "A memoir on abstract geometry" (*ibid.*, vol. 160 (1869) [*ibid.* p. 456]), where speaking of the importance which *abstract geometry, i. e.*, of n dimensions, should have, he says: "The science presents itself in two ways, as a legitimate extension of the ordinary two- and three-dimensional geometries; and as a need in these geometries and in analysis generally." The same idea of the application of hyperspace to infinite systems of plane curves (or of surfaces) is explained clearly with some examples by Halphen at the end of his "Recherches de géométrie à n dimensions" (*Bulletin Soc. Math. de France*,

It is clear that, in this way of thinking, the geometry of hyperspaces no longer presents any novelty of idea; it enters into ordinary geometry, in the treatment of the spreads of forms which occur therein.

Finally we may regard space of n dimensions as defined in the same way as ordinary space, except that we omit the postulate concerning the three dimensions, and consequently modify some of those referring to the straight line and plane. Then the points of the hyperspace are points of the same nature as those which we think of in ordinary space, and are no longer purely analytic forms or geometric forms of various kinds.* To this point of view are committed in particular those who have discussed the foundations of geometry by determining what postulates of the ordinary science can be dispensed with without too much inconvenience. And it is just here that a confusion has frequently made itself apparent with the (physical or philosophical, but not mathematical) question concerning the number of dimensions which ordinary (physical) space *actually* possesses (see the words of Klein previously quoted in a footnote).†

vol. 2 (1873)); it may further be found in the memoir by Clifford, "On the classification of loci," *Phil. Trans.*, vol. 169 (1878) [Math. Papers, p. 305] (in which the *points* of a hyperspace are defined and interpreted in the applications as elements of any nature whatever, curves, surfaces, complexes, etc.); and in several other recent works, as for instance in that of Study, "Ueber die Geometrie der Kegelschnitte, insbesondere deren Charakteristikenproblem," *Math. Annalen*, vol. 27 (1886).

* Cf. Veronese, *loc. cit.*, and "La superficie omaloide normale," etc. (*Memorie Lincei*, 3d series, vol. 19), the note at the bottom of the 2d and 3d pages. The fundamental hypothesis (which is there laid down) for this method of procedure, *i. e.*, that outside of the straight line, of the plane, of ordinary space, \dots , there are always points (which, connected with spaces not passing through them, give rise to higher spaces) is allowable in mathematics, because it does not contradict any of the preceding postulates (after the above mentioned modifications have been made in the ordinary form). [Cf. also Veronese, *Fondamenti di Geometria a più Dimensioni*, (Padova, 1891), or the German translation (Leipzig, 1894). The same notion of hyperspace is the basis of Schoute, *Mehrdimensionale Geometrie* (Leipzig, 1902).]

† [In regard to the *foundations of geometry*, the books by Pasch and by Peano, and since the publication of this article the book by Veronese and the papers of Pieri, Hilbert and others have led mathematicians in recent years more and more to consider geometry from an *abstract*, purely *logical* or *deductive*, point of view, detaching it entirely from every physical consideration. With this tendency, the *primitive* or *undefined* concepts, such as for example *point* or *movement*, come not at all to correspond in any way to determinate physical entities; they may receive instead many different interpretations. Thus in the *primitive* propositions or *postulates* also there exists an analogous liberty. Recourse may still be had to *spatial intuition*

Every one of these ways of considering the linear space in geometry has some advantages; and in particular it may be noted that the first is very general and very simple, but analytical, whereas the third is geometrical and entirely intuitive; and the second is the one which lends itself more readily to numerous applications in ordinary space. But this distinction of points of view does not imply different treatments of (projective) geometry of n dimensions. Indeed for the mathematician this distinction has no real importance. He may well avoid it entirely, and work in hyperspace without making a fixed choice between the three definitions; and he may with perfect right retain all of them in order to have a greater variety of representation and interpretation of his results.

IX.

From what has been said it is evident that the geometry of n dimensions has no mathematical characteristics essentially different from those of ordinary geometry. Spaces of 4, 5, . . . dimensions, as we have defined them, exist for the mathematician precisely in the same way as space of 3 dimensions exists; and they may be studied by the same methods. Thus an algebraic spread M_k (of order m) in an S_n is an analytic or geometric form existing in the same sense as exist the n algebraic functions of k independent parameters by means of which it is defined, or even as there exists an ∞^k algebraic system (of index m) of plane curves or surfaces within an ∞^n linear system, etc.

It would moreover be absurd to say that the figures of hyperspace are of less importance (mathematically) than those of ordinary space. Can any one claim that plane geometry is more important than solid, or that the theory of functions of one or two variables is more important than that of any number n of variables, or that the study of an ∞^3 of forms is of more value than that of a manifold of any order of infinity ∞^n ?

for guidance or as a means of research; not so however for the demonstrations; these must be exclusively *deductions* by *pure logic*.

Following this method the points of a space of 4, or 5, . . . dimensions are treated as above stated in the same way as those of S_3 , the system of postulates being slightly modified. The problem which I have proposed to students in this part of my article: "to lay down a system of independent postulates which will completely define linear space of n dimensions, so that therefrom may be deduced the representation of the points of this space by means of coordinates" was immediately discussed by two of those who were attending my lectures: Amodeo and Fano. Cf. also Killing, *Einführung in die Grundlagen der Geometrie*, Paderborn, 1893 and 1898.]

Naturally however we shall not say that an investigation, *merely because it has reference to hyperspace*, is more important * than an investigation relating to ordinary space ! The comparison is possible only when the first investigation is reduced to the second, taking $n = 3$; and then there is no doubt that to greater generality will correspond greater importance. On the other hand if we compare the *whole* content of geometry of n dimensions with the *whole* of ordinary geometry, we may say the latter is contained as a particular case in the former ; but we should add that all the propositions of the former must be contained in substance in the ordinary geometry, as properties relating to ∞^n systems of ordinary geometric forms.

Now, to the remarks on various tendencies in research which we made first with reference only to ordinary geometry, but which we must think of as extended immediately to geometry of n dimensions, it is proper to add a distinction which naturally arises between the investigations to which the introduction of hyperspace has led. Of the infinite series of new forms and new problems for study which this has brought to light, we find that some are derived by analogy and generalization from those of ordinary space, whereas others arise from concepts which can not be found in the latter. And thus we encounter investigations which are relatively easy to carry on because in them the analogy to ordinary space serves as a guide to the generalization of the methods and results ; but we meet also problems which contain really new difficulties. In order that the edifice (whose construction is only just begun) may be finished, we must undertake both classes of problems.

Consequently it is a mistake to hold that geometry of hyperspace is nothing more than an easily made extension of ordinary geometry ; just as if the whole geometry of ordinary space were a simple and easy generalization of plane geometry ! And those geometers are in the wrong who (to avoid difficulties !) expressly confine themselves to the first class of investigations, and by so doing create the above impression among their readers. It is indeed true that, thanks to hyperspace, those who love easy reasonings have seen fit to increase their number in an extraordinary way. How many new generalizations have they not seen fit to make, so easy that they could

* Nor more difficult ! Experience shows that especially for young students it is fairly easy to become accustomed to the constructions and processes of reasoning relating to geometry of n dimensions.

be announced offhand! What new special forms have they not constructed! What particular transformations, what projections to space of lower* dimensions could they not make them undergo! But here again I must repeat that, though we must not without further consideration condemn a body of reasoning merely because it is easy, nevertheless we must not allow ourselves to be lured on merely by easiness; and we should take care that every problem which we choose for attack, whether it be easy or difficult, shall have a useful purpose, that it shall contribute in some measure to the up-building of the great edifice; that it shall not be one of those simple exercises, those useless generalizations, which are so easy to imagine and to carry out, and which, as they can never become an essential part of the science, so never deserve any sincere praise!

X.

Unlimited space, without any restriction on the number of dimensions, is the medium in which nowadays geometric forms should be considered. By doing this, we obtain besides the great advantage of generality, the further advantage of removing the obstacles which until recently prevented a geometer, aware of the utility for plane geometry of certain stereometric considerations, from seeking in higher space analogous applications for use in ordinary space.† In hyperspace there are no limits; every space is contained in a higher one; and in the latter we may seek for forms which will simplify the study of given forms in the former, by producing them as projections, or as sections, or as apparent outlines, etc. It is here that we find one of the important applications of the geometry of hyperspace to that of ordinary space;‡ but not here alone. We may see at once from the second defini-

* And often neglecting the *inverse problem*, i. e., to see when the form of lower space may be truly regarded as the projection of the form of the higher space!

† Moreover (I may well repeat it) the scientific importance of the geometry of hyperspace would still exist perfectly even without these applications.

‡ It seems that the first instance of such an application is due to Cayley, who in the note: "Sur quelques théorèmes de la géométrie de position," *Crelle's Journal*, vol. 31 (1845), p. 218 [Coll. Math. Papers, vol. 1, p. 321], observes that certain configurations of ordinary space or of the plane may be obtained as sections of the configuration determined by any number of points in any hyperspace. Other applications were then noticed by other mathematicians; but the work which by the use of projection from higher spaces into ordinary space has truly created an epoch, is that of Veronese.

tion given above of higher spaces, that all results relating to these may be translated in numerous ways into ordinary geometric propositions, according to what geometric forms of ordinary space may be taken to replace the points of the hyperspace.* Besides, it is often necessary to turn to higher spaces to find the most convenient representation of certain kinds of manifolds. Thus the study of a linear system of curves in a plane or of surfaces of ordinary space (from the standpoint of the birational transformations) finds a convenient image in the (projective) study of the hyperspatial surface or spread which is represented thereby. For the

Against this method the following objection has been raised. If a form of ordinary space is studied by regarding it as the projection of one belonging to a higher space, we thus introduce into our reasoning a new postulate — the existence of points outside of ordinary space — and therefore the results obtained will not have quite the same value that they would have if they had been obtained by deductions made without leaving ordinary space and the postulates relating thereto. This objection which appears valid if the hyperspaces have been defined by the third method given above, is on the contrary removed at once if recourse be had to the other two. Let us consider a representation of the ordinary points on an ∞^3 spread consisting either of sets of values of three numbers, or of conveniently chosen ordinary geometric forms. This can be done *without any new postulates*. Then let us consider (and this again is possible without adding any new postulates) this ∞^3 spread as being contained in an ∞^n spread (analytic or geometric), and to the latter let us refer our hyperspace reasonings, from which we draw conclusions concerning the ∞^3 spread. Finally we will return to ordinary space by means of the representation of the latter on the ∞^3 spread.

Thus, if with the three coordinates x_1, x_2, x_3 of an ordinary point we associate $n - 3$ zeros, we may represent this very point as an *analytic* point $(x_1, x_2, x_3, 0, \dots, 0)$ of an analytic space of n dimensions, whose points have the coordinates $(x_1, x_2, x_3, \dots, x_n)$.

Or again, to every point x of ordinary space considered as an envelope of planes ξ whose equations are $\xi_x = 0$, let us join a fixed envelope of planes of class $m - 1$, $\phi(\xi) = 0$. We shall obtain an envelope of class m , $\xi_x \cdot \phi(\xi) = 0$, which, by changing the point x , will describe a linear system of ∞^3 such envelopes. By taking m sufficiently large, this system may always be regarded as contained in an ∞^n linear system and then we may take this as the representation of the hyperspace S_n used in our reasoning. The projections and sections, which for simplicity and brevity of statement are made to refer to the points of an S_n , could as well be translated immediately into constructions relating to the envelopes of this linear system, and would thus lead to the same results in ordinary space.

*That geometer is surely not familiar with the variety of interpretations of which the geometry of hyperspace is capable, who recently fell into error in regard to some of the simplest problems concerning ∞^1 systems of plane curves or surfaces of index m , whose solution is contained implicitly in well-known theorems on curves of order m in hyperspace; or who regarded as new the determination of the ruled surfaces of given degree and maximum deficiency, whereas this may be considered as contained in the determination of curves (viz., the M_2^4 of straight lines) of given order and maximum deficiency.

geometry on an algebraic curve of genus p , and particularly for the study of the linear series of point groups on this curve, it is of advantage to turn to certain curves of the various spaces, which represent these linear series, and particularly to the normal curve of order $2p - 2$ in the space S_{p-1} .* Analogous remarks may be made concerning the geometry on a surface, and more generally concerning the geometry on any algebraic spread.† Moreover in these the use of hyperspace will be necessary to represent in a real field the distribution of the points, real and imaginary, of the surface (or spread), *i. e.*, of the values of an algebraic function of *two* (or more) variables, in the same way that in the geometry on a curve, *i. e.*, in case of algebraic functions of *one* variable, recourse has been had to Riemann surfaces. The investigation of the *analysis situs* in hyperspace will therefore find applications of the greatest importance in the geometry on algebraic spreads, *i. e.*, in the theory of algebraic functions of two or more variables.

In this way, making use on the one hand of the methods of transformation of modern geometry, of the considerations of hyperspace, etc., and on the other hand of the concepts due to analysis, the forms which we consider will present themselves under such a variety of aspects and points of view, and will be capable of transformation in so many ways that, while the instruments of study grow ever more numerous, at the same time the interest and importance of the investigations become far greater and acquire besides a singular elegance and charm. Thus the algebraic ∞^1 form (*algebraisches Gebilde*) of genus p is represented by an algebraic curve in any space, or by a ruled surface, or by a ∞^1 system of curves or surfaces, etc.; or by a class of algebraic functions or of abelian functions; or by a system of automorphic functions; or by a Riemann

* In this connection I would like to recall that in the recent Vorlesungen über die Theorie der elliptischen Modulfunctionen by Klein (and Fricke), (vol. 1, Leipzig, 1890; III. Abschnitt) the geometric representation of the algebraic ∞^1 form and of the theory of the algebraic functions and their integrals, and thence of the linear series of point groups, etc., is given precisely by means of curves of all spaces (instead of merely the plane curve, as was customary for a long time). [This example has since been followed in other analytic treatises.]

† I have frequently made mention of the geometry on an algebraic spread, curve, surface, etc., because I wish to call the attention of students to it; it is indeed a subject of prime importance for the whole of mathematics — for analysis as well as geometry — and it would be well if many directed their energy to it!

surface, or by a sphere with p handles, or with p holes, or by a plane curvilinear polygon generating a fuchsian group,* etc., and all these things are equivalent; by means of geometric or analytic transformation, by continuous deformation, by conformal representation, they change one into the other; and from all these representations the study of the algebraic form may derive help.

Those students who, following these suggestions, succeed in obtaining possession of the most varied instruments of research and in comprehending the science from all possible points of view, will attain therefrom the most brilliant success.

TURIN, *February*, 1891.

REPLY TO PROFESSOR SNYDER'S REVIEW OF STUDY'S GEOMETRIE DER DYNAMEN.

IN the January number of the BULLETIN Professor V. Snyder, of Cornell University, lavishes praise and censure on the author of the book under his review, using vigorous terms in both respects. I beg leave to raise, in the present instance, some objections to this mode of criticism.

Preceding reviewers have successfully tried to convey to their readers, space permitting, an idea of the actual contents of the book in question.† Dr. Snyder takes a way of his own. He confines himself mainly to a consideration of the fundamental conceptions of the author's line geometry, as developed in the second half of the volume, and to an enumeration of subjects treated therein. While the latter is very incomplete, omitting, *e. g.*, the most important part of the book (the appendix, referring to kinematics), the former reproduces, not so much the contents of the book itself as those of a paper by Professor E. Müller, and besides develops certain ideas dealing with non-euclidean geometry. By these means Professor Snyder endeavors to make the author's treatment "easier to understand."

* In particular for the elliptic form ($p=1$), the annular surface, the parallelogram, the (2, 2) correspondence between two variables, etc.

† Such reviews have been published by Professor Zindler, of Innsbruck, (*Monatshefte für Mathematik und Physik*, 1903—5 pp.), Professor Wirtinger, of Vienna, (*Zeitschrift für Mathematik und Physik*, vol. 49 (1903)—4 pp.), Professor Daniele, of Pavia, (*Bolletino di bibliografia ecc.*, Dic. 1903—14 pp.).