$$\begin{array}{ccccc} \infty & 0 & 1 & i & i^2, \\ \infty & 0 & i & i^2 & 1, \\ \infty & 0 & i^2 & 1 & i, \end{array} \qquad \begin{array}{ccccc} \infty & 1 & 0 & i^2 & i, \\ \infty & 1 & i & 0 & i^2, \\ \infty & i & 0 & 1 & i^2. \end{array}$$

For $p^n = 5$, the only group solution is one equivalent to that by C. H. Judson in the *American Mathematical Monthly*, 1900, page 72. For $p^n = 7$ there are exactly four group solutions; for $p^n = 9$, there are four. The second group H_{72} actually furnishes solutions.

15. The second paper by Professor Dickson relates to the question of redundancies in Professor Moore's set of relations defining G for the $GF[p^n]$. It is an addition to a paper with the same title in the *Proceedings of the London Mathematical Society*, volume 35 (1902), pages 292–305, which treated the cases for which $p^n \leqq 48$. In the present paper, the theorem of the paper cited is proved, by fortunate devices, to hold for the two new cases $p^n = 3^5$ and $p^n = 5^3$.

EVANSTON, ILL.,
   April 15, 1904.

THOMAS F. HOLGATE,
Secretary of the Section.

THE HEINE–BOREL THEOREM.

BY DR. OSWALD VEBLEN.

This note has for its main object a proof that the Heine-Borel theorem is equivalent, as a continuity axiom, to the Dedekind cut proposition.[*] In §3 the equivalence of the closure of a set with what may be called the H-B property is shown to apply not only to the continuum but to any set of numbers.[†] This result applies in manifolds of any number of dimensions as well as to linear sets.

1. I. *If every number of an interval AB belongs to at least one segment σ of a set of segments $[\sigma]$, then there exists a finite sub-*

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[*] The equivalence in question suggested itself to Mr. N. J. Lennes and myself while we were working over some elementary propositions in real function theory.

[†] This extends a theorem of Dr. W. H. Young to the effect that every closed linear set has the H-B property. Cf. W. H. Young, "Overlapping Intervals." *Proceedings of the London Mathematical Society*, p. 384, vol. 35 (1903).

set of $[\sigma]$, σ_1, σ_2, σ_3, $\cdots$, σ_n, such that every number of AB belongs to at least one of the segments σ_1, σ_2, $\cdots \sigma_n$.

By a *segment* $AB = \sigma$ $(A < B)$ is meant the set of real numbers $[X]$ such that $A < X < B$. By an *interval* AB is meant the set of real numbers $[X]$ such that $A \le X \le B$. Proposition I is essentially involved in Heine's [*] proof of the theorem of uniform continuity, but has first been stated in explicit form by Borel, [†] who was careful to specify that the set of segments of which he spoke should be at most numerably infinite. The theorem, however, is valid whatever be the set $[\sigma]$, and indeed to the restricted form given it by Borel the remarks below do not wholly apply.

In its generalized form, the Heine-Borel theorem can be used to replace the tedious "caging" process which appears so frequently in rigorous analysis. As an example, let us take Weierstrass's theorem that any infinite set of numbers $[X]$ lying on a bounded interval AB has at least one limit number. If the theorem should fail, then about each number of the interval AB there would be a segment σ containing not more than one number of the set $[X]$. By the Heine-Borel theorem, there would exist a finite subset $\sigma_1, \sigma_2, \cdots, \sigma_n$ of these segments σ such that every number of AB would lie on at least one such segment. But as each of $\sigma_1, \sigma_2, \cdots, \sigma_n$ contained only one number of $[X]$, $[X]$ could contain only a finite number of numbers, contrary to hypothesis.

2. Besides being of great convenience in the routine of limiting processes, the generalized form of the Heine-Borel theorem is equally fundamental with the Dedekind cut proposition as an axiom of continuity for the real number system. The Dedekind statement is as follows:

II. *If all the numbers of an interval AB fall into two classes* $[M]$ *and* $[N]$, *such that every M is greater than every N, then there exists a number O such that $M \ge O \ge N$ for every M and every N.*

The assertion that I and II are equally fundamental may be more exactly stated as follows:

III. *Admitting the ordinal relations of the real number system, I is a consequence of II and II is a consequence of I.*

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[*] Heine, "Die Elemente der Functionenlehre," *Crelle*, vol. 74 (1872), p. 188.

[†] Borel, "Sur quelques points de la théorie des fonctions," *Annales de l'École Normale Supérieure*, 3d series, vol. 12 (1895), p. 51.

The proof that I is a consequence of II is well-known * and is given in a generalized form below. To prove II from I, observe that if there is no number O of the kind required, $[M]$ has no greatest lower bound and $[N]$ no least upper bound. Therefore, about each number M there is a segment not including any number N, and about each number N a segment not including any number M. Every number of AB being either an M or an N, this set of segments satisfies the hypothesis of I. Hence there is a finite set of segments σ_1, σ_2, $\cdots$, σ_n including every number of AB and each including only numbers of $[M]$ or only numbers of $[N]$. Let A be the smaller of A and B. Of the finite number of segments which include A, no one includes or is bounded by a number of $[M]$. Let the largest bound of the segments including A be A'. Proceed with A' (which is a number of $[N]$) as with A, obtaining A'' of $[N]$ and so on. Since there are a finite number of segments σ_1, $\cdots$, σ_n, the process leads by a finite number of steps to the conclusion that none of the segments include numbers of $[M]$, contrary to hypothesis.

3. The Heine-Borel theorem has a generalization of some importance in the theory of sets. Let us say that a set of real numbers $[X]$ has the H-B property if, from every set of segments $[\sigma]$ such that every number X belongs to at least one σ, there can be selected a finite subset σ_1, σ_2, $\ldots$, σ_n such that every X belongs to at least one σ_k ($k = 1, 2, \ldots, n$). By a *closed* set, let us mean one that includes all its limit numbers.

IV. *A necessary and sufficient condition that a bounded set $[X]$ of numbers shall have the H-B property is that $[X]$ is closed.*

It is evident that an infinite set not including its limit numbers has not the H-B property. For if $[X]$ lies on AB ($A < B$) and has a limit number C, not an X, let $[\underline{C}]$ be the set of all numbers smaller than C, and let $[\overline{C}]$ be the set of all numbers larger than C. The set of segments, $A\underline{C}$, $B\overline{C}$ has no finite subset including every number X.

In proving that a closed set has the H-B property, we assume without loss of generality that $A < B$ and that A and B belong to $[X]$. Either, one of the segments $\sigma = A_0 A_1$ includes AB, in which case $n = 1$, or the ends $[A']$ of the seg-

* Cf. Schoenflies, " Bericht über die Mengenlehre," *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 8 (1899), p. 51.

ments including A are all less than B. It is an immediate and well-known consequence of II that every bounded set of numbers has a least upper bound. Let A_1 be the least upper bound of $[A']$. Either A_1 belongs to the set $[X]$ or, by the closure of $[X]$, one of the segments including A includes all numbers X between A and A_1. If A_1 is an X, we denote it by X_1, if not, let X_1 be the smallest number X (existent because $[X]$ is closed) which is greater than A_1. Proceed with X_1 as with A_1, obtaining A_2 and, if necessary, X_2. By a finite number of steps, we obtain $A_k \geqq B$; for, if not, the bounded set X_1, X_2, $\cdots$ would have a least upper bound X_ω which, because of the closure of $[X]$, would be a limit number of $[X]$. Hence there would be a segment σ including X_ω; this segment must include at least one X_m and, therefore A_{m+1}, contrary to the hypothesis that A_{m+1} is the least upper bound of the ends of the segments including X_m.

To obtain the set $\sigma_1, \cdots, \sigma_n$, let $\sigma_1, \cdots, \sigma_k$ be a set of segments of $[\sigma]$ including X_1, X_2, $\cdots$, X_{k-1}, B; let σ_{k+1} be the segment of $[\sigma]$ including A and all numbers of $[X]$ between A and X_1 in case $A_1 \neq X_1$ and a segment of $[\sigma]$ including A and the lower end of σ_1 in case $A_1 = X_1$; similarly, let $\sigma_{k+h}(h = 2, \cdots, k)$ include X_{h-1} and all numbers between X_{h-1} and X_h if $X_h \neq A_h$ or include X_{h-1} and the lower end of σ_h if $A_h = X_h$. $n = 2k$ (unless $n = 1$ as above).

4. The theorems proved above depend upon the linear arrangement and continuity of the real number system and not at all upon the operations of addition and multiplication. They therefore apply equally well in the geometry of a straight line, even without metric hypotheses, and in fact to any interval of a continuous curve without multiple points.

The validity of the Heine-Borel theorem for the plane (or for space of any number of dimensions) suggests that it can be used as a continuity axiom, not only for real numbers, but also for complex number systems.

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