WHITTAKER'S MODERN ANALYSIS.

A Course of Modern Analysis. By E. T. WHITTAKER. Cambridge, England, University Press, 1902. xvi + 378 pp.

This book, as the subtitle explains, is intended as "an introduction to the general theory of infinite series and of analytic functions; with an account of the principal transcendental functions." There is certainly room for a book of this sort, especially as the transcendental functions treated are of the most varied kinds, including the gamma function, hypergeometric functions with the special and limiting cases of Legendre's and Bessel's functions, as well as elliptic functions, both Weierstrass's and Jacobi's normal forms being treated in detail. Somewhat more than half the book is devoted to an account of these functions. Part I, which to some extent may be regarded as introductory to this second part, is designed, to use the author's own words, to contain "an account of those methods and processes of higher mathematical analysis which seem to be of greatest importance at the present time." There is so much of interest and importance in these chapters, parts of which can be found nowhere else in the English language, that it may seem ungracious to criticise the choice of material which has commended itself to the author. There are, however, one or two matters of fundamental importance whose omission it is hard to justify. The subject of series, which claims the lion's share, forms only one of several infinite processes which occur constantly in analysis and which are, in fact, freely employed in the latter part of the book, though usually without justification except in the case of series. As an illustration of what is meant, attention may be called to the treatment of infinite products. Four pages are devoted to the question of the convergence and absolute convergence of such products, but the conception of their uniform convergence is not touched upon. Or again, to come to a matter of still greater importance, the distinction between finite and infinite definite integrals * is not

* I follow T. J. I'A. Bromwich (*Proc. London Math. Soc.* Ser. 2, vol. 1, p. 176 where reference to G. H. Hardy is made) in translating the German terms *eigentliches Integral* and *uneigentliches Integral* by *finite integral* and *infinite integral* respectively. The terms *proper* and *improper* integrals, which have sometimes been used, although they come nearer to the German, do not seem

brought out, nor is it explained what precautions must be used in working with infinite integrals. Thus, for example, the question as to whether we have the right to assume that

$$\frac{d}{dx}\int_c^\infty \phi(t, x)dt = \int_c^\infty \frac{\partial\phi(t, x)}{\partial x}\, dt,$$

is a question precisely on a par, both in its internal nature and in its practical importance, with the question whether we have the right to differentiate a series term by term. Mr. Whittaker considers this latter question with care; the former he does not even mention, although he frequently has occasion to use infinite integrals in the latter part of his book. These three subjects — series, infinite products, and infinite integrals — are merely special cases, and by no means the only important special cases, of an infinite process of the sort indicated by the equation

$$F(x) = \lim_{\alpha=k} f(x,\alpha)$$

where k may be finite or infinite, and α may or may not be restricted to a discrete set of values in approaching k. Thus we have the case of a series if in the above formula we let $k = +\infty$, restrict α to positive integral values, and let $f(x, \alpha)$ denote the sum of the first α terms of the series. Or, again, we obtain the case of an infinite integral by letting

$$f(x, \alpha) = \int_c^\alpha \phi(t, x)dx, \qquad k = +\infty.$$

Now a fundamental question is whether $f(x, \alpha)$ approaches its limit $F(x)$ *uniformly* throughout a certain interval, that is whether, after choosing our limit of error ϵ small at pleasure, we can then push our limiting process so far that, from this point on, $f(x, \alpha)$ differs from its limit $F(x)$ at all points of the interval by less than ϵ. If this is the case, we say that our infinite process converges uniformly, whether this process be expressed

desirable from the point of view of pure English. It will be noticed that the terms finite and infinite, as here used, correspond precisely to the use of these words in finite and infinite products, finite sums and infinite series, etc. *i. e.*, the infinite integral is the limit of a finite integral. The term infinite integral is intended to apply not merely to the case mentioned in the text in which one of the limits of integration is infinite, but also to the case in which the function becomes infinite at one end of the interval.

as a series, as an infinite product,* or as an infinite integral. When once the idea of the uniform convergence of series has been grasped and its applications understood, it would be a matter of only a very few pages to treat the subject from the much more general standpoint just indicated.

Before leaving this subject of uniform convergence we may perhaps be permitted to express the opinion that, especially in a treatise devoted largely to analytic functions of a complex variable, Weierstrass's theorem should not have been omitted which states † that if the terms of a series are functions of a complex variable which throughout a certain two-dimensional region are analytic, and if throughout this region the series converges uniformly, it represents an analytic function whose derivatives of all orders can be found by differentiating the series term by term. This theorem is, in fact, tacitly assumed in the second part of the book.

If we have thus indicated a number of unfortunate omissions, it is only right to add that other important matters of great interest are included which have not yet made their way into the ordinary treatises. Among these may be particularly mentioned infinite determinants and asymptotic developments.

Turning now to the second part of the book in which the more important transcendental functions are treated, we find a large amount of detailed information which cannot fail to prove valuable to the student who wishes to penetrate into the theory of one or the other of these functions. Although general points of view are only lightly touched upon in these chapters they are by no means omitted. The author might, however, have made his treatise more valuable by going a little farther in this direction. Thus, to mention a small matter, Gauss's formula for expressing the logarithmic derivative of the gamma function as a definite integral (p. 186) really holds a unique position among the numerous definite integral formulæ connected with the gamma function in that the integrand is single valued instead of being infinitely multiple valued; an advantage well known to Gauss, as is evident from his remarks in a letter to Bessel in an analogous case. A line or two of text would have made this fact clear. Again, Weierstrass's

* A slight deviation of phraseology is probably desirable for infinite products so that such a product which at one or more points of the interval "diverges" to the value zero shall not be called uniformly convergent.

† This theorem can be at once extended to other infinite processes.

theorem that the gamma function satisfies no algebraic differential equation might well have been mentioned, with a reference to Hölder's proof, as this theorem gives the student a much clearer idea of the place among the other transcendental functions which this function occupies. We mention these two points merely as samples of the way in which by an incidental remark now and then the reader's horizon might be enlarged.

While the book before us is still far from attaining the standard of rigor which French, German, and Italian writers now regard as almost a matter of course, it is a gratifying sign of progress to find in an English book such an attempt at rigorous treatment as is here made. It is only fair to the reader, however, to warn him that he should be on the lookout for pitfalls, for they abound in this subject, and the author has not always succeeded in avoiding them.

The examples which Mr. Whittaker, following the custom of English writers, has introduced in large number into this treatise cannot fail to be of great service in making the subject a living one to many readers, or rather students, of his book. The solution (by the student himself be it understood, not, after the approved German fashion, by the writer of the treatise) of a problem, well chosen to illustrate a theoretical point, often does more to clarify the theory and impress it on the student's mind than any amount of direct study of the principle concerned. While the advantage of such problem solving decreases as the student becomes more mature, it by no means disappears even in the case of most full fledged mathematicians; and many a person well beyond his student years will find that he can spend profitable hours in sharpening his wits on the whetstone with which he is here supplied in the shape of problems.

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HARVARD UNIVERSITY,
January, 1904.