the preceding case there is precisely one such $\overline{\kappa\lambda}$-segment $\overline{P_0P_1}$ inscribed in the $\overline{\alpha\beta}$-arch $\overline{AC}$.

The case $0 > \kappa > \beta$ is reduced to the preceding case $\alpha > \kappa > 0$, by the transformation R of § 1, 8°. For under R a $\kappa\lambda$-segment P_0P_1 transforms into a $\overline{\kappa\lambda}$-segment $\overline{P_1P_0}$, and an $\alpha\beta$-arch AB transforms into an $\overline{\alpha\beta}$-arch $\overline{BA}$, where

$$\overline{\kappa} = -\kappa, \quad \overline{\lambda} = \lambda; \quad \overline{\alpha} = -\beta, \quad \overline{\beta} = -\alpha,$$

so that indeed $\overline{\alpha} > \overline{\kappa} > 0$.

THE UNIVERSITY OF CHICAGO,
February 6, 1904.

THE RICCATI DIFFERENTIAL EQUATIONS WHICH REPRESENT ISOTHERMAL SYSTEMS.

BY DR. EDWARD KASNER.

THE object of this note is to determine the isothermal systems of (plane) curves which can be represented by an equation of Riccati type

(1) $$y' = P + Qy + Ry^2,$$

where P, Q, R denote arbitrary functions of x. The result obtained is that there are four distinct systems of this kind, namely, those given by the equations (3), (4), (6), (7) below.

It has been shown by Lie [*] that the differential equation of any isothermal system is of the form

$$y' = \tan F(x, y)$$

where F is a harmonic function; that is,

$$F_{xx} + F_{yy} = 0.$$

By a simple transformation this result may be expressed more conveniently for the present purpose as follows :

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[*] Lie-Scheffers, Differentialgleichungen, 1891, p. 157.

In order that an equation $y' = f(x, y)$ shall represent an isothermal system, it is necessary and sufficient that the function f satisfy the partial differential equation

$$(2) \qquad (1 + f^2)(f_{xx} + f_{yy}) - 2f(f_x^2 + f_y^2) = 0.$$

It will be convenient to divide the discussion of the question proposed into three cases: In the first case $R = 0$ and $Q = 0$, so that the equation (1) does not involve y; in the second, $R = 0$, but $Q \neq 0$, so that the equation is linear; in the third, $R \neq 0$, so that the equation is quadratic in y.

Case I.   Equations of Type $y' = P$.

Substituting P for f in the general condition (2), we find that P must satisfy the (ordinary) equation of second order

$$(1 + P^2)P' - 2PP'^2 = 0,$$

the solution of which is found to be

$$P = \tan(ax + b).$$

Hence the only differential equations of type I representing isothermal systems are $y' = \tan(ax + b)$.

As long as a does not vanish, this equation may be reduced, by means of a change of origin together with a change of unit distance, to $y' = \tan x$; the corresponding system of curves is

$$(3) \qquad\qquad y + \log \cos x = \text{const.}$$

When a vanishes, on the other hand, the equation may be reduced to $y' = 0$, and the corresponding system is simply a set of parallel lines

$$(4) \qquad\qquad\qquad y = \text{const.}$$

Since the equations of type I are characterized by the fact that they admit the one-parameter group of translations in the y direction, the result obtained may be stated:

If an isothermal system admits a group of translations, it may be reduced to one of the forms (4) or (3).

Case II.   Equations of Type $y' = P + Qy$, where $Q \neq 0$.

Substituting $P + Qy$ for f in the condition (2), and equating coefficients of powers of y to zero, we find the following sys-

tem of equations for the determination of the two functions P and Q:

$$Q^2 Q'' - 2QQ'^2 = 0,$$

(5)
$$2(QQ'' - Q'^2)P - 4QQ'P' + Q^2 P'' = 0,$$
$$Q''(1 + P^2) + 2QPP'' - 2Q(P'^2 + Q^2) - 4Q'PP' = 0,$$
$$(1 + P^2)P'' - 2P(P'^2 + Q^2) = 0.$$

By assumption, $Q \neq 0$, so that the first equation may be written $QQ'' - 2Q'^2 = 0$. The integral of this is

$$Q = \frac{1}{ax + b}.$$

Without loss of generality we may assume $Q = ax^{-1}$, this being equivalent to merely a change in the axis of ordinates. Substituting this value of Q in the second equation of (5), we find

$$x^2 P'' + 4xP' + 2P = 0,$$

so that P is necessarily of the form $P = Ax^{-2} + Bx^{-1}$.

The corresponding linear equation $y' = P + Qy$ is

$$y' = Ax^{-2} + Bx^{-1} + ax^{-1}y = \frac{a\left(y + \dfrac{B}{a}\right)x + A}{x^2};$$

which by a proper change in the axis of abscissas may be reduced to

$$y' = \beta x^{-2} + ax^{-1}y.$$

The corresponding values of P and Q are

$$P = \beta x^{-2}, \quad Q = ax^{-1}.$$

Finally, the last two equations of the set (5) give the additional conditions $\beta = 0$, $a = \pm 1$. The corresponding linear differential equations are $y' = y/x$ and $y' = -y/x$, the solutions of which are

(6) $$y/x = \text{const.,}$$

(7) $$xy = \text{const.}$$

The only isothermal systems whose differential equation is of type II *are reducible to the systems* (6) *or* (7), *the former representing a pencil of straight lines and the latter a system of equilateral hyperbolas with common asymptotes.*

Since types I and II make up the class of linear equations :

If the differential equation of an isothermal system is linear, the system is of one of forms (3), (4), (6), (7).

Among the linear equations, the class $y' = Qy$ form a special class characterized by the property of invariance with respect to the group of affine transformations

(8) $x_1 = x, \quad y_1 = \lambda y.$

Of the systems obtained, (4), (6) and (7) belong to this subclass.

The only isothermal systems which allow the one parameter group of affine transformations (8) *are the systems of straight lines* (4), (6) *and the system of equilateral hyperbolas* (7).

Case III.   *Equations* $y' = P + Qy + Ry^2$, *where* $R \neq 0$.

In this case the discussion shows that no solutions exist. Applying condition (2), we find

(9)
$$\{R''y^2 + Q''y + P'' + 2R\}\{1 + (Ry^2 + Qy + P)^2\}$$
$$- 2\{Ry^2 + Qy + P\}\{(R'y^2 + Q'y + P')^2$$
$$+ (2Ry + Q)^2\} = 0.$$

By equating the coefficients of the various powers of y in the first member to zero, a system of seven differential equations of the second order is obtained for the determination of the three unknown functions P, Q, R.   To prove that these equations, under the assumption that R does not vanish, are inconsistent, it will be sufficient to consider four of them, namely, those obtained from the sixth, fifth, fourth and second powers of y.

The first equation is

$$R^2R'' - 2RR'^2 = 0.$$

The complete solution of this equation is $(ax + b)^{-1}$, but without loss of generality we may assume

(10) $R = ax^{-1} \quad (a \neq 0).$

The second equation, obtained from the coefficient of y^5 in (9), is

$$R^2 Q'' - 4R' R' Q + 2(RR'' - R'^2)\, Q = 0,$$

which reduces to

$$x^2\, Q'' + 4x Q' + 2Q = 0$$

after the substitution of the value of R given in (10). Integrating, we find that Q must be of the form $Ax^{-2} + Bx^{-1}$.

From the discussion up to this point, it follows that if an isothermal system of type III exists, its equation must be of the form

$$y' = \frac{\alpha x y^2 + (Bx + A)y}{x^2} + P.$$

By the substitution

$$y \sim y + \frac{B}{2\alpha},$$

equivalent to merely a change in the axis of abscissas, the equation reduces to the form

$$y' = \frac{\alpha x y^2 + \beta y}{x^2} + P$$

where β is some constant and P is some function of x (not necessarily the same as that denoted by P in the original equation). We may therefore, without loss of generality, take

(11) $$Q = \beta x^{-2}.$$

The third and fourth equations referred to above, corresponding to the coefficients of y^4 and y^2 in (9), become, after reduction by means of (10) and (11),

(12) $$x^5 P'' + 4x^4 P' + 2x^3 P = 6\alpha x^4 + 2\beta^2 \alpha^{-1}.$$

$$2\alpha x^{-3}(1 + P^2) + 12\beta^2 x^{-6} P + (2\alpha x^{-1} + P'')(\beta^2 x^{-4} + 2\alpha x^{-1} P),$$

(13) $$- 2\alpha x^{-1}(P'^2 + \beta^2 x^{-4}) - 2\beta x^{-2}(4\alpha\beta x^{-3} - 4\beta x^{-3} P')$$

$$- 2P(4\beta^2 x^{-6} + 4\alpha^2 x^{-2} - 2\alpha x^{-2} P') = 0.$$

The integration of (12) gives

$$P = \alpha x + \gamma x^{-1} + \delta x^{-2} - \beta^2 \alpha^{-1} x^{-3}.$$

Substituting this value in (13), we find that the coefficient of the highest power of x in the first member is $- 2\alpha^3$. This however cannot vanish on account of the assumption that $\alpha \neq 0$. It follows that there are no functions satisfying condition (9) except when $R = 0$. Therefore

No equation of type III, *that is of the Riccati type proper, can represent an isothermal system.*

The complete answer to the question proposed at the beginning of the paper may now be stated :

If the differential equation of an isothermal system is included in the form $y' = P + Qy + Ry^2$, then the system must belong to one of four species: 1°, a set of parallel straight lines; 2°, a pencil of straight lines; 3°, a system of equilateral hyperbolas with common asymptotes; 4°, a system of logarithmic cosine curves $y + \log \cos x = const.$, which may also be written $e^y \cos x = const.$

COLUMBIA UNIVERSITY,
    December, 1903.

ON SOME PROPERTIES OF GROUPS WHOSE ORDERS ARE POWERS OF A PRIME.

BY DR. W. B. FITE.

(Read before the American Mathematical Society, December 28, 1903.)

IN the *Transactions,* volume 3 (1902), page 334, the writer has shown that if a metabelian group of order p^m, where p is a prime, contains an abelian subgroup of order p^{m-a}, the p^a power of every operator is invariant. In the same article, page 349, it was shown that a metabelian group of odd order cannot be a group of cogredient isomorphisms if it has a set of generators such that the order of any one of them is not a divisor of the least common multiple of the orders of all the others. This latter was generalized somewhat in the BULLETIN, 2d series, volume 9 (1902), page 140. It is the purpose of the present article to carry this generalization somewhat further and to show that the first theorem is a special form of a theorem applicable to all groups whose orders are powers of a prime.

Let G be a group of order p^m and class k that contains an abelian subgroup G_1 of order p^{m-1}. If A is any operator of G_1 and B any operator of G not contained in G_1, we have