THE WINTER MEETING OF THE CHICAGO SECTION.

A REGULAR meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held in the Central High School Building at St. Louis, in connection with the American Association for the Advancement of Science, on Thursday and Friday, December 31, 1903, and January 1, 1904. The following members were present:

Professor C. H. Ashton, Professor H. T. Eddy, Professor G. W. Greenwood, Professor E. R. Hedrick, Professor Thomas F. Holgate, Professor J. A. Miller, Dr. H. L. Rietz, Professor D. A. Rothrock, Mr. Oscar Schmiedel, Professor J. B. Shaw, Professor Ormond Stone, Mr. E. H. Taylor, Professor C. A. Waldo, Professor L. G. Weld, Professor H. S. White, Professor J. N. Van der Vries.

At the first session Professor Ormond Stone presided, Professor Weld occupying the chair during the remainder of the meeting. On the afternoon of the first day a visit was paid to Washington University and the grounds of the St. Louis Exposition, through the kindness of the Commissioners of the Exposition.

The following officers were elected for the ensuing year: Secretary, Professor Thomas F. Holgate; Members of the programme committee, Dr. S. E. Slocum and Dr. G. A. Bliss.

The report of the committee appointed at the meeting of the section held in January, 1902, to consider the requirements for the master's degree for candidates making mathematics their major subject, was taken up for discussion and after some amendments was recommended for publication. The report will appear in the BULLETIN.

The following papers were read:

(1) Professor E. R. HEDRICK: "The law of the mean for functions of several variables."

(2) Professor E. W. DAVIS: "The elliptic functions and the general symmetric group on four letters.

(3) Professor G. A. MILLER: "On the generalization and extension of Sylow's theorem."

(4) Mr. OSCAR SCHMIEDEL: "Analogues of the jacobian identity that involve four elements."

(5) Professor H. S. WHITE: "Linear systems of hyperelliptic plane curves of the first kind."

(6) Professor E. R. HEDRICK: "A class of pseudo-contact transformations."

(7) Mr. J. V. COLLINS: "Some developments in vector analysis."

(8) Professor J. B. SHAW: "Algebras defined by finite groups."

(9) Dr. H. L. RIETZ: "Groups in which certain commutative operations are conjugate; and complete sets of conjugate operations."

(10) Professor L. E. DICKSON: "A generalization of symmetric and skew-symmetric determinants."

(11) Professor Jacob WESTLUND: "Primitive roots of an ideal in an algebraic number field."

(12) DR. G. A. BLISS: "An existence theorem for a differential equation of the second order, with an application to the calculus of variations."

(13) DR. SAUL EPSTEEN: "The definition of a reducible hypercomplex number system."

(14) PROFESSOR L. E. DICKSON: "Memoirs on abelian transformations."

(15) MR. H. E. JORDAN: "Group characters of the linear fractional and binary linear homogeneous groups of determinant unity in any Galois field."

(16) MR. H. E. JORDAN: "Group characters of the group of all linear fractional substitutions in any Galois field."

(17) MR. J. J. QUINN: "A linkage for describing the conic sections by continuous motion."

(18) MR. T. R. RUNNING: "Circles represented by the equation $u^3P + Lu^2Q + MuR + NS = 0$."

The papers of Professor Miller, Mr. Quinn, and Mr. Running were carried over from the programme of Section A of the American Association. The first was presented by Professor Weld, the last two were read by title. Abstracts of these papers were included in Professor Weld's report of the meeting of Section A, in the March BULLETIN, pages 291–293. Professor Davis's paper was presented by Professor White, and that of Mr. Collins by Professor Shaw. In the absence of the authors the papers by Professor Dickson, Professor Westlund, Dr. Bliss, Dr. Epsteen and Mr. Jordan were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. The law of the mean for functions of one variable is known to hold, even when the derivative of the function is discontinuous. For functions of two variables, the law of the mean is usually stated for functions which have continuous derivatives, and the common form when the derivatives are discontinuous involves two undetermined numbers θ, θ'. In Professor Hedrick's paper a form is derived which holds when the function has discontinuous derivatives, and which involves only one undetermined number θ. Taking

$$\phi(t) = f(x + ht, y + k) + f(x, y + kt),$$

where $f(x, y)$ is the given function of the two variables x and y, it is easy to show that the derivative of $\phi(t)$ exists, provided the derivatives of $f(x, y)$ merely exist. It follows that the law of the mean for a single variable applies, and hence

$$\phi(1) - \phi(0) = \phi'(\theta), \qquad (0 < \theta < 1).$$

The substitution of the value of $\phi(t)$ in this form gives the desired formula

$$f(x + h, y + k) - f(x, y) = hf_x(x + \theta h, y + k) + kf_y(x, y + \theta k).$$

This formula still involves two distinct points, and these points lie on the sides of the rectangle whose corners are (x, y) and $(x + h, y + k)$. It would seem desirable to find a formula which should involve only one point, and that a point inside the rectangle.

2. Professor Davis's paper starts from the standard formula

$$\int_\alpha^x \frac{dx}{\sqrt{(x - \alpha)(x - \beta)(x - \gamma)(x - \delta)}} = \frac{2}{\sqrt{(\alpha - \gamma)(\beta - \delta)}}$$

$$\operatorname{sn}^{-1} \sqrt{\frac{(\beta - \delta)(x - \alpha)}{(\alpha - \delta)(x - \beta)}}, \qquad \operatorname{mod} \sqrt{\frac{(\beta - \gamma)(\alpha - \delta)}{(\alpha - \gamma)(\beta - \delta)}}$$

and classifies the various changes produced by the permutations of α, β, γ and δ. This is made easy and interesting by the use of Glaisher's notation *sc*, *cs*, *sd*, *ds*, etc.

4. Mr. Schmiedel presented some identities involving four elements, analogous to the jacobian identity with three. If Uf is the symbol of infinitesimal transformation, the jacobian identity is written

$$[(U_1 U_2) U_3] + [(U_2 U_3) U_1] + [(U_3 U_1) U_2] = 0;$$

or, retaining indexes only, $[(12)3] + [(23)1] + [(31)2] = 0$; or, in abbreviated form, $(1, 2, 3) = 0$. Now the analogues with four elements consisting of terms derived from each other by cyclic permutation are $(1, 2, 3, 4) = [(13)(24)]$, together with twenty-three others obtained from this by interchange of figures. If the sum in the first member is to consist of terms derivable from each other by advancing each element by unity, another set of twenty-four identities is obtained of which the first member is this sum, and the second $\pm[(13)(24)]$, or twice this, according to the permutation.

5. Of hyperelliptic curves in a plane there are two principal classes : (1) Those whose passage through one point does not involve passage through a definite corresponding point and (2) systems in which there does occur such a pairing of points. In a linear system of the first kind the adjoints are degenerate and are equivalent to sets of right lines ; hence the system is reducible by Cremona transformations to the curves of order m having a common multiple point of order $m - 2$. (This theorem is found to be due to Castelnuovo, 1890.) Systems of the second kind are "involution curves" of an involution of order 2, and can be generated from rational systems by transformations (due to Clebsch and Noether) of the three types which relate a double plane of deficiency zero to a simple plane. In his paper Professor White shows typical reduced systems for odd deficiency.

6. An ordinary contact transformation may be defined as a transformation of all the direction elements in a plane (or in space), which carries any curve (or surface in space) into a curve (or surface). Restricting ourselves, for simplicity, to the plane, it is evident that when we are dealing with a specialized family of curves only, more general transformations must exist, which carry the curves of that peculiar family into curves. These transformations will be called *pseudo-contact transformations*. In Professor Hedrick's second paper it is shown that,

given *any* transformation $X = f(x, y, y')$, $Y = \phi(x, y, y')$, of the direction elements of one plane into the points of another, a pseudo-contact transformation is determined, which carries the solutions of a certain differential equation into curves. This transformation is useful in the study of this particular family of curves, and in particular, in the study of this differential equation. Conversely, given any differential equation of the first order, or any one-parameter family of curves, there exist an infinite number of such pseudo-contact transformations which are useful in the study of the differential equation, or of the family. Examples of such transformations are given in the paper, and the extension to space is pointed out.

7. Mr. Collins showed how not only quaternions, but also other similar branches of vector analysis can be developed quite simply from certain fundamental laws.

8. Professor Shaw's paper develops the general method of reducing any abstract group, when considered as a linear associative algebra, to its expression in canonical form.* The general form of any expression in an algebra of this kind is $\Sigma a_{rs0}\,\lambda_{rs0}$. The units λ_{rs0} fall into p sets, the tth set containing μ_t^2 forms, so that, n being the order of the group,

$$n = \mu_1^2 + \mu_2^2 + \cdots + \mu_p^2.$$

The general theory is then applied to several examples. The forms of the canonical units are developed for abelian groups, dihedral groups, the symmetric group on three letters, the tetrahedral group, and the octahedral group. In conclusion, the very intimate connection between this theory and Frobenius' theory of group characters and group determinants, and Dickson's theory of groups defined for an abstract field, is discussed. Also the place of this development in the general theory of linear associative algebra is fixed.

9. In Dr. Rietz's paper it is first shown that the group of order two and the symmetric group of order six are the only groups in which every two commutative operations, neither of which is identity, belong to the same conjugate set of operations. The nature of the groups is next examined, which have the property

* *Transactions*, vol. 4 (1903), pp. 251–287.

that any two commutative operations of the same order belong to the same conjugate set. It is shown, among other things, that these groups are either perfect groups or are isomorphic to the group of order 2^α all of whose operations except identity are of order 2, and that in the latter case the commutator subgroup corresponds to identity in the quotient group of order 2^α. If, in the group under consideration, all operations of the same order are conjugate, α is not greater than unity.

A closely related problem is then considered. Inquiry is made as to what groups exist in which the number λ of complete sets of conjugate operations, and the number n of distinct primes in the order are connected by a certain simple relation. No groups are possible with $\lambda < n + 1$. The group of order 2 and the symmetric group of order 6 are the only groups with $\lambda = n + 1$. The group of order 3, the non-cyclic group of order 10, the alternating group of order 12, and the simple group of order 60 are the only groups with $\lambda = n + 2$.

10. Professor Dickson's first paper considers the determinant $A = |\alpha_{ij}|$ where α_{ji} and α_{ij} are conjugate imaginaries for every $i, j = 1, \cdots, m$. It follows readily that A has a real value. Set $\alpha_{rs} = a_{rs} + iA_{rs}$. For $m = 3$, $A - |a_{rs}|$ is a ternary quadratic form in A_{12}, A_{13}, A_{23}, whose determinant equals $|a_{rs}|$. For $m = 4$, $A - |a_{rs}| - S^2$ is a quadratic form in the six quantities A_{rs} whose determinant is the second compound of $|a_{rs}|$, where S is the Pfaffian $A_{12}A_{34} - A_{13}A_{24} + A_{14}A_{23}$. An interpretation in line coördinates is thus evident.

The chief claim for attention to the symmetrically conjugate determinant A is the elegance with which it expresses one of the concomitants (Begleitform) of the quadratic form $\Sigma a_{rs}x_rx_s$.

The paper appeared in the December number of the *American Mathematical Monthly*.

11. If A be a prime ideal in an arbitrary algebraic number field, there always exist primitive roots of A. In Professor Westlund's paper the question whether there exist primitive roots of any ideal is discussed, and the necessary and sufficient conditions for the existence of such primitive roots are determined.

12. Picard has shown that two points p and q can be joined by a uniquely determined solution of the equation

$$\frac{d^2y}{dx^2}=f\left(x,\ y,\ \frac{dy}{dx}\right),$$

provided that q lies in a suitably chosen region adjoining p. Dr. Bliss discusses the nature of this solution considered as a function of the coördinates of p and q, and applies the results to prove an existence theorem for the equation

$$\frac{1}{\rho}=G\left(x,\ y,\ \frac{dx}{dt},\ \frac{dy}{dt}\right),$$

where G is a positively homogeneous function of dx/dt and dy/dt, and ρ is the radius of curvature of the solution $x = x(t)$, $y = y\ (t)$. This theorem is in turn applied to show that if the two points p and q are near enough to each other, they can always be joined by a uniquely determined extremal of the calculus of variations problem, corresponding to the integral

$$I=\int_{t_0}^{t_1}F\left(x,\ y,\ \frac{dx}{dt},\ \frac{dy}{dt}\right)dt,$$

and this extremal minimizes I between p and q.

13. According to Benjamin Peirce (*American Journal*, volume 4, page 100) and Scheffers (*Mathematische Annalen*, volume 39, page 317), a hypercomplex number system

$$E \equiv E_jE_k \equiv l_l \cdots l_m l_{m+l} \cdots l_n$$

is reducible when the following conditions are fulfilled: C_1, E_j forms a system by itself; C_2, E_k forms a system by itself; $A, l_jl_k = 0$; $B, l_kl_j = 0$ $(j = l, \cdots m;\quad k = m + 1, \cdots n)$.

The chief result of Dr. Epsteen's paper is to show that for systems containing a modulus the conditions C_1, C_2 are consequences of A, B; hence, a hypercomplex number system, containing a modulus is reducible if $l_jl_k = l_kl_j = 0$.

The paper will be submitted for publication in the *Transactions*.

14. A complete theory of canonical forms of linear transformations in an arbitrary field has been developed and applied, for example, to the distribution into sets of conjugates of the transformations of the general linear group. When we pass

from the general to a particular linear group, we encounter an incomparably more difficult problem. For the quaternary abelian group in a Galois field, an investigation was given in the *Transactions*, volume 2 (1901), pages 103–138. Professor Dickson's second paper aims at a systematic method for the $2m$-ary abelian group in a general field. The results are complete for $m = 1, 2, 3$, and are applied to the distribution of the operators into sets of conjugates.

In view of the holoedric isomorphism of the senary abelian group G, modulo 2, with the group of the 28 bitangents to a quartic without double points, the author gives (as an illustration of the results of the memoir) the following table showing the period of any transformation T of a chosen complete set of conjugates and the number of transformations of G commutative with T:

Period.	Commutative.	Period.	Commutative.	Period.	Commutative.
1	all	4	$2^6.3$	6	$2^4.3^2$
2	$2^7.3$	4	2^7	7	7
2	$2^9.3$	4	$2^7.3$	8	2^4
2	$2^9.3^2$	5	2.3.5	8	2^4
2	$2^9.3^2.5$	6	$2^2.3$	9	3^2
3	$2^2.3^3$	6	$2^2.3^2$	10	2.5
3	$2^3.3^4$	6	$2^2.3^2$	12	$2^2.3$
3	$2^4.3^3.5$	6	$2^3.3^2$	12	$2^3.3$
4	2^5	6	$2^4.3$	12	$2^3.3$
4	$2^6.3$	6	$2^4.3^2$	15	3.5

The paper will appear in the *American Journal of Mathematics*.

15, 16. In Mr. Jordan's papers the group characters for each of the three systems of groups are determined. They are about as simple as those obtained by Frobenius (*Berliner Sitzungsberichte*, 1896, pages 985–1021) for the linear fractional group of determinant unity modulo p. The case $p^n = 2^n$ requires separate treatment. Interesting relations hold between the characters of a group and those of one of its quotient groups.

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