THE APRIL MEETING OF THE SAN FRANCISCO SECTION.

THE third regular meeting of the San Francisco Section of the AMERICAN MATHEMATICAL SOCIETY was held on Saturday, April 25, 1903, at Stanford University. The following fifteen members were present:

Professor R. E. Allardice, Dr. E. M. Blake, Professor H. F. Blichfeldt, Professor G. C. Edwards, Professor R. L. Green, Professor L. M. Hoskins, Dr. D. N. Lehmer, Dr. J. H. McDonald, Professor G. A. Miller, Dr. H. C. Moreno, Dr. T. M. Putnam, Professor Irving Stringham, Mr. L. C. Walker, Mr. A. W. Whitney, Professor E. J. Wilczynski.

A morning and an afternoon session were held, Professor Stringham acting as chairman at both sessions. Professor M. W. Haskell was elected a member of the programme committee in place of Professor E. J. Wilczynski, who will be abroad next year.

The following papers were read at this meeting:

(1) Professor E. J. WILCZYNSKI: "Invariants of systems of linear partial differential equations, and the theory of congruences."

(2) Dr. D. N. LEHMER: "Preliminary report on a table of smallest divisors."

(3) Professor H. F. BLICHFELDT: "Note on linear substitution groups of finite order."

(4) Professor R. E. ALLARDICE: "On some curves connected with a system of similar conics through three points."

(5) Dr. SAUL EPSTEEN: "Necessary and sufficient condition for the existence of invariant subgroups."

(6) Professor G. A. MILLER: "On reciprocal groups."

(7) Dr. H. C. MORENO and Professor G. A. MILLER: "On the non-abelian groups in which every subgroup is abelian."

(8) Mr. W. A. MANNING: "On the class of primitive substitution groups."

(9) Miss Ida M. SCHOTTENFELS: "Generational definition of an abstract group simply isomorphic with the simple substitution group G_{20160}^{21}."

(10) Dr. T. M. PUTNAM: "Certain subgroups of the quater-

nary linear fractional group of determinant unity, in the general Galois field."

The papers by Dr. Epsteen and Miss Schottenfels were read by Professor Wilczynski and the secretary respectively. All the other papers were read by their authors. Abstracts are given below.

1. Professor Wilczynski considers in his paper a system of two linear homogeneous partial differential equations with two independent variables x_1 and x_2

$$(1) \qquad \begin{aligned} \Omega &= ay_1 + by_2 + cz_1 + dz_2 + ey + fz = 0, \\ \Omega' &= a'y_1 + b'y_2 + c'z_1 + d'z_2 + e'y + f'z = 0, \end{aligned}$$

where $y_k = \partial y/\partial x_k$, $z_k = \partial z/\partial x_k$, $(k = 1, 2)$.

If one makes the transformations

$$(2) \quad y = \alpha\eta + \beta\zeta, \qquad z = \gamma\eta + \delta\zeta, \qquad (\alpha\delta - \beta\gamma \neq 0);$$

$$(3) \quad \xi_1 = f_1(x_1, x_2) \qquad \xi_2 = f_2(x_1, x_2), \qquad \left(\frac{\partial(\xi_1, \xi_2)}{\partial(x_1, x_2)} \neq 0\right),$$

where α, β, γ, δ, f_1 and f_2 are arbitrary functions of x_1 and x_2, (1) is transformed into another system of the same form. Moreover one may replace the system by

$$(4) \quad \phi\Omega + \psi\Omega' = 0, \quad \chi\Omega + \omega\Omega' = 0, \quad \phi\omega - \psi\chi \neq 0,$$

where ϕ, ψ, χ, ω are arbitrary functions of x_1 and x_2, and obtain again a system of the same form. These are the most general transformations which convert a general system (1) into another of the same form.

The author considers the functions of the coefficients of (1) which remain invariant under these transformations, and actually computes them for a reduced form.

If one considers four sets of simultaneous solutions

$$y^{(k)} = g^{(k)}(x_1, x_2) \quad z^{(k)} = h^{(k)}(x_1, x_2), \quad (k = 1, 2, 3, 4)$$

one may interpret them as the homogeneous coördinates of two points in space, the locus of each being a surface. The transformations (2), (3) and (4) merely displace these points on the line joining them and otherwise change merely the parametric representation of the two surfaces. They leave invariant, therefore, the congruence of lines obtained by joining corresponding points of the two surfaces. The theory of a system of form (1) is therefore geometrically a theory of congruences. This leads to geometric interpretations for the vanishing of certain invariants, the determination of the developable surfaces of the congruence, its focal surfaces, etc. The properties thus obtained are all projective. But not all projective properties will appear, owing to the fact that congruences which are not mere projective transformations of each other, may belong to the same system (1). This question is investigated in the paper. All systems of form (1) may be reduced to a few simple types. The general type corresponds to a congruence with distinct non-degenerate focal surfaces. For instance, the system of partial differential equations of the theory of functions belongs to a congruence whose focal surfaces have degenerated into two distinct curves, and may be taken as the type of such systems.

The theory of the Laplace transformations of an equation of the form

$$\frac{\partial^2\theta}{\partial x \partial y} + a\frac{\partial\theta}{\partial x} + b\frac{\partial\theta}{\partial y} + c\theta = 0,$$

which is so important in the theory of surfaces, is also generalized to a system of the kind considered and finds its geometrical interpretation the same as in that older theory. It is of course connected with the theory of the covariants of such a system.

The author points out moreover that this theory can be generalized for other systems, and that geometrically other configurations, complexes, etc., may be studied in this way.

2. Dr. Lehmer's factor tables are intended to give the smallest divisors of all numbers less than ten millions. Multiples of 2, 3, 5 and 7 are not tabulated. All other numbers are of the form $210x + R$ where R is prime to 210. There are 48 values of R which give the headings for as many columns. The values of x are listed at the side. Each page of the tables contains 100 rows and 48 columns and serves to factor 21,000 numbers. The plan of the table is somewhat similar to that used by Lebesgue (Tables diverses pour la décomposition des nombres en leurs facteurs premiers. Par V. A. Lebesgue, Paris, 1864), but differs from it in several important particulars.

3. In determining the linear homogeneous substitution groups of finite order in two variables, Gordan makes use of the equation $1 + \cos \phi_1 + \cos \phi_2 + \cos \phi_3 = 0$; ϕ_1, ϕ_2, ϕ_3 being rational angles ("Ueber endliche Gruppen linearer Transformationen einer Veränderlichen," *Mathematische Annalen*, 1877, page 29). By forming the corresponding equation for a linear homogeneous group G in n variables it may be proved that all the substitutions of such a group whose orders are products of primes greater than $(n - 1)(2n + 1)$ form a subgroup G_1. Professor Blichfeldt proved that G_1 is abelian and self-conjugate under G. He also pointed out the following results: (1) No simple group whose order is divisible by a prime greater than $(n - 1)(2n + 1)$ can be simply isomorphic with a linear homogeneous substitution group in n variables. (2) If the determinants of the substitution of a linear homogeneous substitution group in n variables are all unity, and if n is a prime, then the order of the group is not divisible by any prime greater than $(n - 1)(2n + 1)$.

4. This paper may be considered a continuation of the one published in the *Transactions*, volume 4 (1903), page 103. In it Professor Allardice gives an investigation of the loci of the vertices and foci of the system of conics through three points. He also considers the envelope of the directrices. Each pair of vertices and each pair of foci has a distinct locus, and each directrix has a distinct envelope. The loci are of the eighth and sixth degrees respectively, and the envelope is of the fourth degree.

5. Dr. Epsteen shows that a certain theorem of Lie's may be stated as follows: The necessary and sufficient conditions that a continuous r-parameter group G_r may contain an invariant subgroup g_s $(s < r)$, is that the adjoint group G_r shall be reducible. A number of consequences are easily deduced from this by utilizing the notions of reducibility of the adjoint group. For instance, if G_r is integrable, so is its adjoint group. If a group coincides with its adjoint, and contains an invariant subgroup, it is reducible. The quotient groups are the same for all composition series apart from their order. By introducing a notion, which in the theory of continuous group corresponds to the group of contragredient isomorphisms, the "total adjoint group" is defined, and the total adjoint group is then shown to be always reducible.

6. Several years ago Professor Tanner called attention * to the indications of the existence of a reciprocity theorem, viz., that in every group of totitives of order g there are as many subgroups of order g_1 as there are of order $g \div g_1$, g_1 being any divisor of g. As the groups of totitives are included among the abelian groups,† the theorem whose existence Professor Tanner suspected is included in the theorem that every abelian group of order g has as many distinct subgroups of order g_1 as there are subgroups of order $g \div g_1$. The last theorem is readily proved by means of the characteristics of an abelian group. Professor Miller proved this theorem without using characteristics and pointed out some of its applications.

7. At the first meeting of the Section, Dr. Moreno presented some results in regard to non-abelian groups in which every subgroup is abelian.‡ The present paper completes the determination of all such groups. The following are some of the most important results: The order of such a group cannot be divided by more than two distinct primes. If it is divisible by two primes (p, q), the Sylow subgroup whose order is a power of one of these primes is of type $(1, 1, 1, \cdots)$, while the other Sylow subgroup is cyclic. The former of these Sylow subgroups is also the commutator subgroup and includes no invariant operator besides the identity. When the order of such a group is p^a, it contains just $p + 1$ subgroups of order p^{a-1} and none of these can have more than three invariants. The paper will be offered to the *Transactions* for publication..

8. Mr. Manning investigated the primitive substitution groups of class $3p$, p being a prime, which contain substitutions of order p and degree $3p$. Jordan announced that the degree of a primitive group containg such a substitution cannot exceed $3p + 4$. By using this theorem it is shown that such groups do not exist when $p > 5$. When $p = 5$, there are the three primitive groups, G_{80}^{16}, G_{240}^{16}, and G_{4080}^{17}. The two cases, in which $p = 2$ and $p = 3$, are not here considered since they have already been worked out by Jordan.

* Tanner, *Proc. London. Math. Soc.*, vol. 27 (1896), p. 329.

† *Annals of Math.*, vol. 2 (1900), p. 79.

‡ BULLETIN, vol. 8 (1902), p. 434. When P_{m-1} has only two invariants each of them may be unity. This special case was overlooked.

9. In a paper presented to the Society, October, 1901, Miss Schottenfels determined the generational definition of the abstract group of order 960 holoedrically isomorphic to the ternary group Galois field $[2^2]$ of the same order with the following definition :

$$
\begin{aligned}
&E_1^3 = E_2^2 = E_3^2 = B_1^2 = I,\\
(1)\qquad &(E_1E_2)^3 = (E_2E_3)^3 = (E_1E_3)^2 = I,\\
&(B_1E_1) = (B_1E_2)^4 = (B_1E_3)^2 = I,
\end{aligned}
$$

where E_1, E_2, E_3, B_1 satisfying relations (1) are the generators of the group.

This group can be immediately extended by an operator C of order 7, say $C^7 = I$, to the abstract simple group of order $8!/2$ holoedrically isomorphic to the simple ternary group Galois field $[2^2]$ of order $8!/2$, with generators C, E_1, E_2, E_3, B_1, satisfying the further relations

$$
(2)\qquad (CE_1)^5 = (CE_2)^5 = (CE_3)^5 = (CB_1)^5 = I.
$$

10. In the determination of the cyclic subgroups of the quaternary group of determinant unity, taken in a general Galois field, the canonical forms with unequal multipliers give rise to some difficulties which do not present themselves in the others. Dr. Putnam investigates these cases, starting with the homogeneous group and passing from it to the fractional form in the usual way. The paper is supplementary to the article published in the *American Journal of Mathematics*, volume 24 (1902), page 319.

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