THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, April 25, 1903, extending through the usual morning and afternoon sessions. About fifty persons were in attendance, including the following forty members of the Society.

Dr. Grace Andrews, Dr. C. L. Bouton, Professor E. W. Brown, Mr. J. S. Brown, Professor F. N. Cole, Miss E. B. Cowley, Professor E. S. Crawley, Dr. W. S. Dennett, Dr. L. P. Eisenhart, Dr. William Findlay, Professor H. B. Fine, Professor T. S. Fiske, Miss Ida Griffiths, Professor G. H. Hallett, Miss Carrie Hammerslough, Professor James Harkness, Dr. H. E. Hawkes, Dr. J. I. Hutchinson, Dr. S. A. Joffe, Dr. Edward Kasner, Professor C. J. Keyser, Mr. L. L. Locke, Professor James Maclay, Dr. C. R. Mann, Professor E. H. Moore, Professor W. F. Osgood, Dr. I. E. Rabinovitch, Miss I. M. Schottenfels, Dr. Arthur Schultze, Dr. W. M. Strong, Mr. John Tatlock, Jr., Professor H. W. Tyler, Professor E. B. Van Vleck, Professor J. M. Van Vleck, Mr. H. E. Webb, Professor A. G. Webster, Dr. J. K. Whittemore, Miss E. C. Williams, Dr. Ruth G. Wood, Professor R. S. Woodward.

The President of the Society, Professor Thomas S. Fiske, occupied the chair. The Council announced the election of the following persons to membership in the Society: Professor W. N. Ferrin, Pacific University, Forest Grove, Ore.; Mr. E. H. Koch, Jr., Mackenzie School, Dobbs Ferry, N. Y.; Professor N. C. Riggs, Armour Institute of Technology, Chicago, Ill.; Mr. K. D. Swartzel, Harvard University, Cambridge, Mass. Thirteen applications for admission to the Society were received.

The following papers were read at this meeting:

(1) DR. H. E. HAWKES: "On non-quaternion number systems in seven units."

(2) Professor B. O. PEIRCE: "On families of curves which are the lines of certain plane vectors, either solenoidal or lamellar."

(3) Professor E. W. BROWN: "On the variation of the arbitrary and given constants in dynamical equations."

(4) Dr. L. P. EISENHART: "Congruences of tangents to a surface, and derived congruences."

(5) Dr. H. F. STECKER: "Least distance in the non-euclidean plane."

(6) Professor L. E. DICKSON: "Fields whose elements are linear differential equations."

(7) Dr. SAUL EPSTEEN: "On linear differential congruences."

(8) Professor R. S. WOODWARD: "The deviation from the vertical of falling bodies" (preliminary communication).

(9) Dr. EDWARD KASNER: "The automorphic groups of the manifolds defined by a general and a symmetric determinant."

(10) Mr. C. H. SISAM: "On some directrix curves on quintic scrolls."

(11) Mr. L. I. NEIKIRK: "Groups of order p^m which contain cyclic subgroups of order p^{m-3}."

(12) Miss I. M. SCHOTTENFELS: "On the simple groups of order $8!/2$."

(13) Dr. E. B. WILSON: "The so-called foundations of geometry."

Mr. Sisam's paper was communicated to the Society through Dr. Snyder; Dr. Epsteen's through Professor Dickson. Mr. Neikirk was introduced by Professor Crawley. In the absence of the authors, Mr. Sisam's paper was read by Dr. Hutchinson, and the papers of Professor Peirce, Dr. Stecker, Professor Dickson, Dr. Epsteen and Dr. Wilson were read by title.

Professor Dickson's paper will be published in the BULLETIN. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. At the December meeting of the Society Dr. Hawkes gave a method for the enumeration of all non-quaternion number systems with skew units. The present paper applies this method to the case $n = 7$ and deduces expeditious means of carrying out further enumerations should it be desirable.

2. If $u \equiv f(x, y) = c$ represents any family of curves in the xy plane, any vector V the components of which are of the form

$$\left(F(x, y), \; - F(x, y) \cdot \frac{\partial u}{\partial x} \Big/ \frac{\partial u}{\partial y}, \; 0\right)$$

where F is arbitrary, evidently has the u curves for its lines. Of all the vectors which have these given lines, an infinite number are in general solenoidal and an infinite number are lamellar, but no one can be both solenoidal and lamellar, unless u happens to satisfy Lamé's well known condition for an isothermal parameter [$\nabla^2(u)/h_u^2$ a function of u only], in which case there are a set of such vectors connected by a linear relation.

In certain general problems in mathematical physics, classes of lamellar or solenoidal vectors appear which are known to satisfy in each case some general condition common to all of the class. Sometimes, for instance, the tensor, or the divergence, or the tensor of the curl of a vector must be constant in a given region; or the tensor, or the divergence, or the curl must be expressible in terms of the parameter of the lines of the vector, or in terms of the parameter of the orthogonal lines. Professor Peirce's paper discusses briefly the possible lines of plane vectors which satisfy a few of the more common conditions. For example, it is shown that, if $u = c$, $v = k$ are the equations of curves in the xy plane which cut each other orthogonally, the condition that the u curves shall be possible lines of a set of solenoidal vectors, the tensors of which involve u only, is the condition that the v curves shall be possible lines of a set of lamellar vectors, the tensors of which involve u only. If the u curves are possible lines of a solenoidal vector, the tensor of which is expressible in terms of v, the v curves are the lines of a lamellar vector, the tensor of which is a function of v only. If a solenoidal vector which has the u curves for lines has a tensor expressible in terms of u, the u curves must form a parallel system, etc., etc.

3. Professor Brown's paper contains certain results with reference to the arbitrary constants which arise in the integration of a set of dynamical equations and also with reference to the constants which occur in the force function. The equations are supposed to have been solved by trigonometric series with arguments some of which are arbitrary and some of which are present in the force function. In the latter part of the paper it is considered under what circumstances we may substitute a variable value for a constant as well in the solution as in the force function. It then appears that Newcomb's well known theorem on the secular accelerations is a particular case of sub-

stitution of a variable for a constant in the solution of a set of such differential equations.

4. Given a family of curves upon a surface S; the tangents to these curves form a rectilinear congruence for which S is one of the focal sheets. The other focal sheet is a determinate surface S_1, and upon it there is a family of curves to which the lines of the congruence are tangent. If tangents are drawn to the curves on S_1, which are conjugate to the latter, a second congruence is formed with S_1 for one focal sheet and another surface S_2 for the other sheet. This process can be continued indefinitely unless one of the surfaces reduces to a curve. In a similar manner we can get another sequence of congruences by starting with the tangents to the conjugates of the given curves on S. These are the *derived congruences* of Darboux. After finding the expressions for the different functions of these congruences in terms of functions of S, Dr. Eisenhart applies them to a study of such sequences where all the congruences of the series are of a particular form. Thus it is shown that the system of curves upon three consecutive surfaces cannot be formed of lines of curvature, so that there cannot be a sequence of congruences of Guichard, as Bianchi calls those whose developables touch the focal sheets in lines of curvature of the latter. From this it follows almost immediately that there are no series of congruences of which each member is a normal congruence.

If a congruence of Ribaucour is defined as one whose developables cut the mean surface in a conjugate system, the condition that the tangents to the curves $v =$ const. of a surface form such a congruence takes a very simple form. From this it is found that, when the tangents to the curves $u =$ const. conjugate to the former also form a congruence of this kind, all the members of the sequence are congruences of Ribaucour. A large number of surfaces are found to have a conjugate system which has this property. It is shown that the only isothermic surfaces whose lines of curvature form such a system have either of the following linear elements when referred to this system:

$$ds^2 = UV(du^2 + dv^2), \quad ds^2 = e^{UV}(du^2 + dv^2),$$

where U is a function of u alone and V is a function of v alone.

When one proceeds to the determination of the surfaces with these linear elements, he finds that U or V must be constant, so that the only surfaces are surfaces of revolution. From this one is led to the result that the tangents to the meridians of a surface of revolution form a normal congruence of Ribaucour.

The equations of condition that the tangents to a given family of curves on S form a cyclic congruence can be brought to a very simple form, so that quite a number of theorems follow almost immediately. Among others one notes that when the tangents to the lines of curvature in one system upon a surface form a cyclic congruence, it is at the same time a congruence of Ribaucour and conversely. The discussion closes with an inquiry as to the existence of a sequence of cyclic congruences for which the circles of each congruence are equal, and it is found that there are none of this kind.

5. The problem treated in Dr. Stecker's paper is an application of the calculus of variations to determine, directly from the distance integral, the least distance between two fixed points in the non-euclidean plane.

7. In the *Comptes Rendus*, 1897, page 489, Guldberg shows that it is possible to find for linear differential forms a theory which is analogous to the Galois field theory. In the present paper Dr. Epsteen develops this theory up to the point of demonstrating the theorem: Every existent differential field of order s may be represented as a Guldberg field of order $s = p^n$; the $GuF(p^n)$ is defined uniquely by its order; in particular, it is independent of the special irreducible differential congruence used in its construction. This is the exact analogue of a well-known theorem of E. H. Moore (Chicago Congress Papers, page 220; Dickson, Linear Groups, page 14).

8. In his preliminary communication Professor R. S. Woodward reviewed the history of the problem of the deviation from the vertical of falling bodies from the time of its earliest treatment by Gauss * and Laplace,† in 1803, down to the present time. He also presented a solution of the problem differing from any of the solutions hitherto published, but giving the same results, to terms of the order of the square of the angular velocity of the earth, as those derived by Gauss, and

* *Werke*, vol. 5. pp. 495–503.

† *Mécanique Céleste*, vol. 4, chapter 5.

as those required by, though not fully worked out in, Laplace's investigation.

9. The first part of Dr. Kasner's paper is devoted to certain general principles of invariant theory, while the second is an application to the problem announced in the title. If the coefficients of a form are interpreted as homogeneous point coördinates in a space of sufficiently high dimension, then any invariant equated to zero represents a variety which is transformed into itself by a certain group of collineations. This group includes the so-called induced group of the coefficients, and in many cases coincides with it. It follows that any covariant (in the domain of the coefficients) of an invariant is also an invariant, a principle which includes for example the Aronhold process as a very special case.

The conclusion concerning the general determinant

$$\Delta \equiv | a_{ik} | = 0$$

is that its collineation group is composed of two continuous systems containing $2(n^2 - 1)$ parameters. The proof involves, in addition to the general principles, a geometric discussion of the manifold $\Delta = 0$ in space of $n^2 - 1$ dimension, in particular its two-fold projective generation. The case $n = 2$ gives of course the well known six parameter group connected with a quadric surface.

10. Mr. Sisam's paper proved the two following theorems: 1° Every unicursal quintic scroll has three coplanar generators; the residual may be either a conic, a double directrix or a simple directrix and a fourth generator. 2° If the asymptotic lines of a (4, 1) scroll are reducible, two real and two imaginary generators issue from each real point of the four-fold line.

11. The groups of operations of order p^m which contain a cyclic subgroup of order p^{m-2} have been determined by Miller (see *Transactions*, volume 3, number 4). In the present paper Mr. Neikirk has determined the groups of order p^m which contain a cyclic subgroup of order p^{m-3}. The investigations are number-theoretic in character and are based on a division of the groups considered into classes. This division is shown to depend on the partitions of m which give the orders of the independent operations that enter into the con-

struction of the required group. Denoting the cyclic subgroup of order p^{m-3} by $\{P\}$ and the partition by $[a, b, \cdots]$, the numbers of resulting groups are given by: $[m-3, 3]$, 12 groups, 4 in which $\{P\}$ is self-conjugate, including the abelian group of this type, and 8 in which $\{P\}$ is not self-conjugate; $[m-3, 2, 1]$, $20+p$ groups in $8+p$ of which $\{P\}$ is self-conjugate; $[m-3, 1, 1, 1]$, 5 groups, in all of which $\{P\}$ is self-conjugate. The total number of these groups is therefore $37+p$.

The formal generational equations of all these types in terms of independent operators are given in the course of the investigation.

12. Miss Schottenfels's paper contains the proof that: 1° All types of simple groups of order $8!/2$ or 20160, contain at least 960 conjugate subgroups of order 7. 2° No simple group of this order contains an operator where period is a multiple of 7 greater than 1.7, and all such simple groups are identical, with respect to the number of such operators or elements.

13. The first part of Dr. Wilson's paper contains criticisms on Professor Hilbert's "Grundlagen der Geometrie" which appeared in the *Mathematische Annalen*, October, 1902. The second part is occupied with the construction of some Hilbertian geometries and in particular a two-dimensional geometry in three dimensions. The paper is to appear in the *Archiv der Mathemathik und Physik.*

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